network robustness

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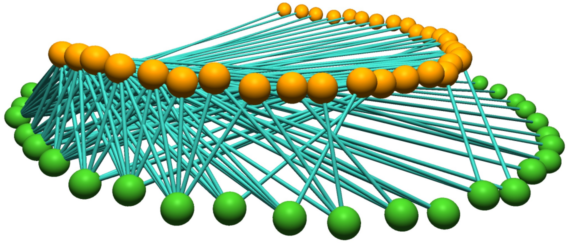
why should we care about network robustness?

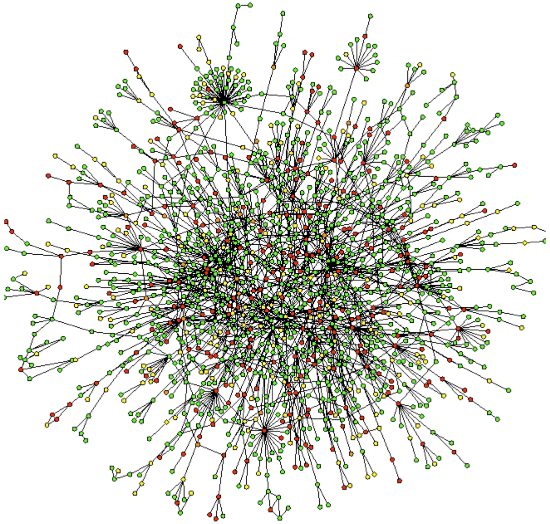
ecological networks



1. network robustness and percolation transition

criminal network

sexual contacts



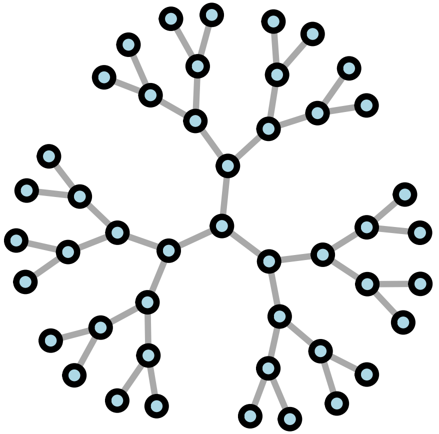
2. Molloy-Reed criterion

3. what is relevant for ecology according to a physicist

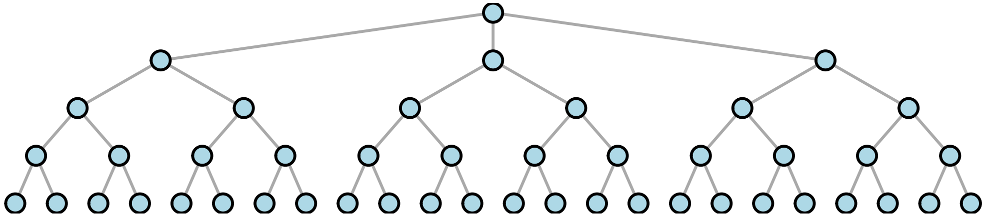
network robustness
and
percolation transition

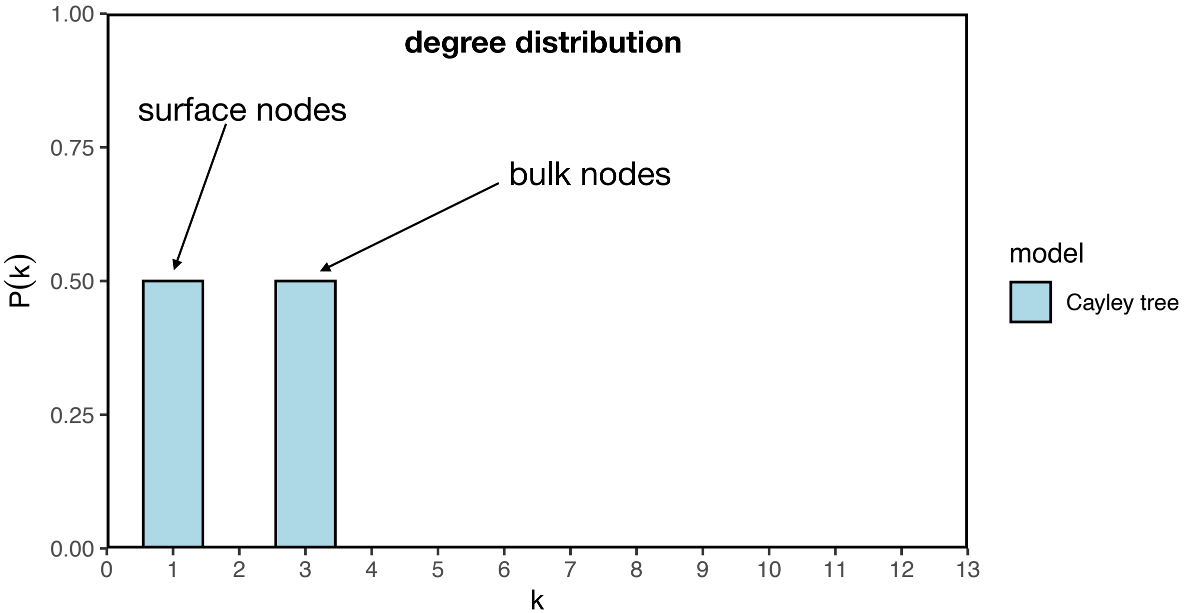
Cayley tree with k=3

star-like representation



tree-like representation





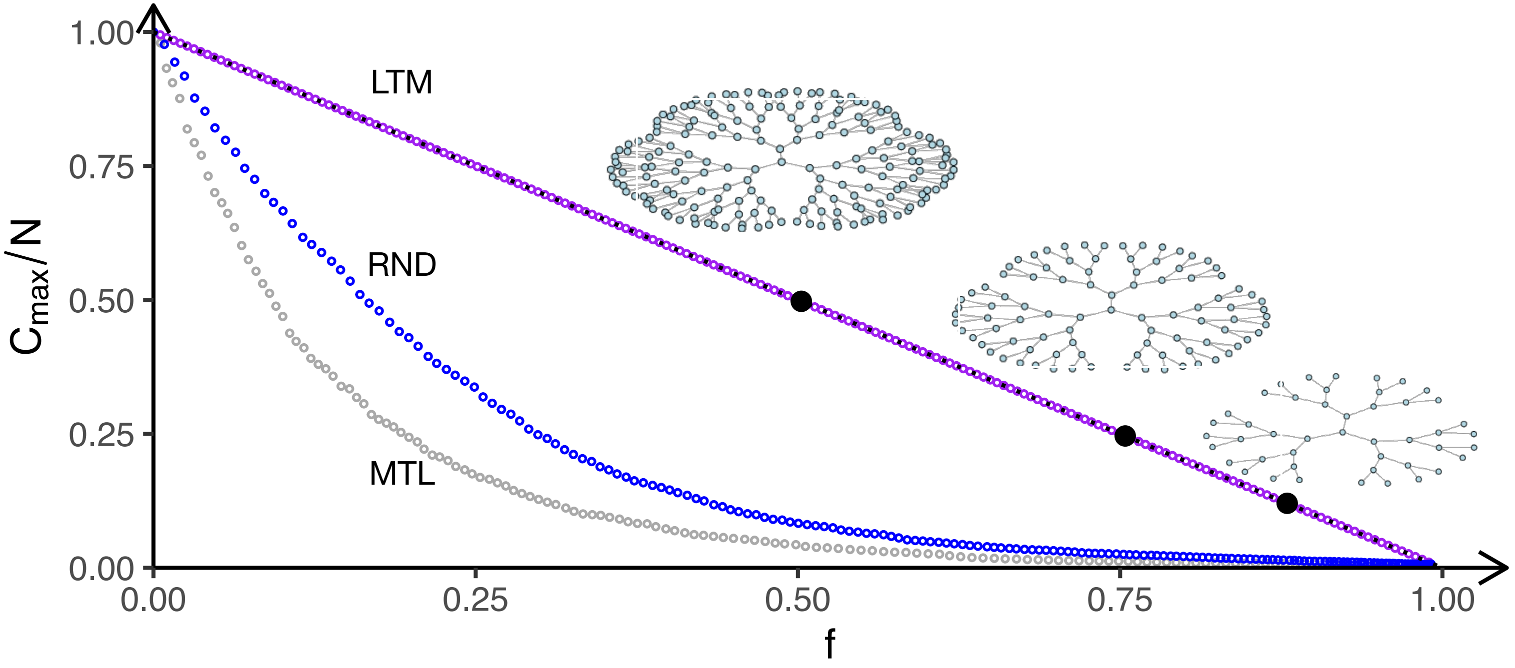
k = number of links per node

Cayley tree

RND: random removal of nodes

MTL: from most connected to least connected

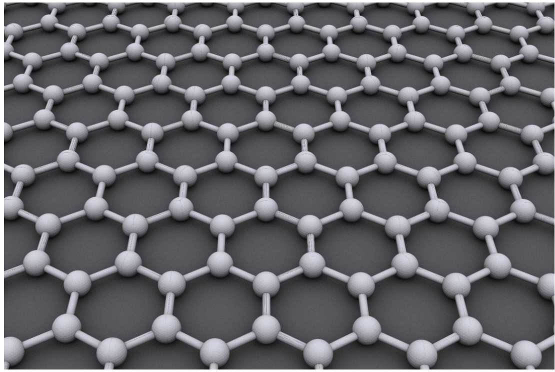
LTM: from least connected to most connected

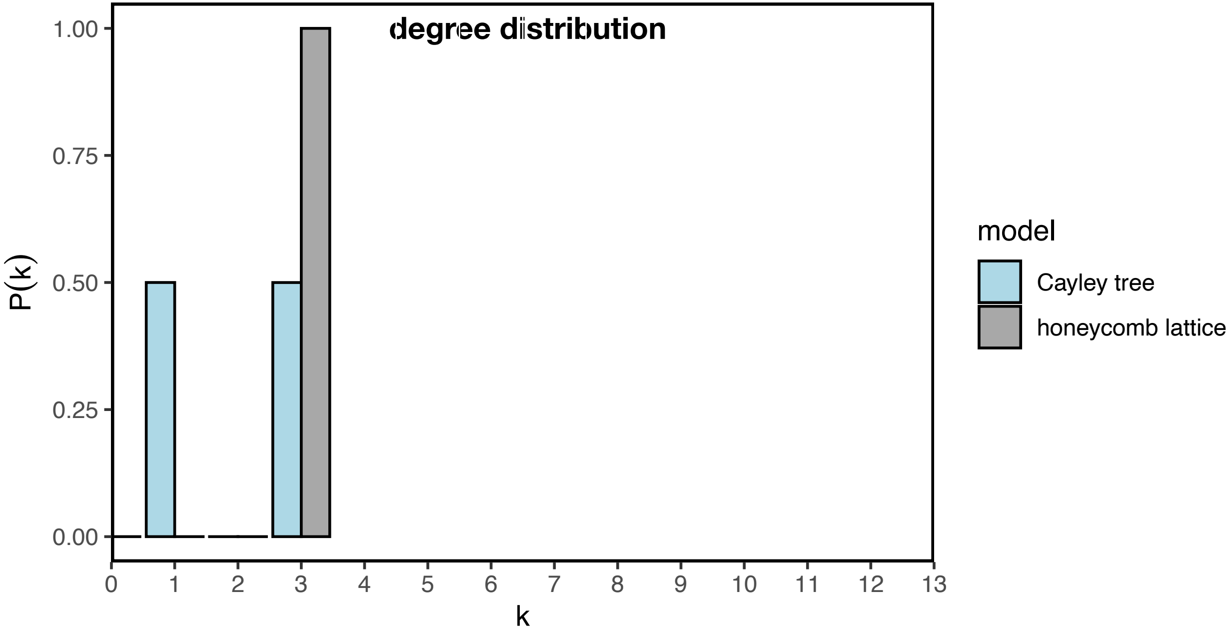


honeycomb lattice

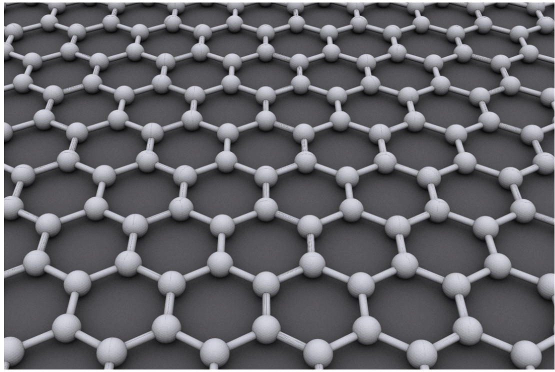
named after bees…

… but we think of Graphene





honeycomb lattice



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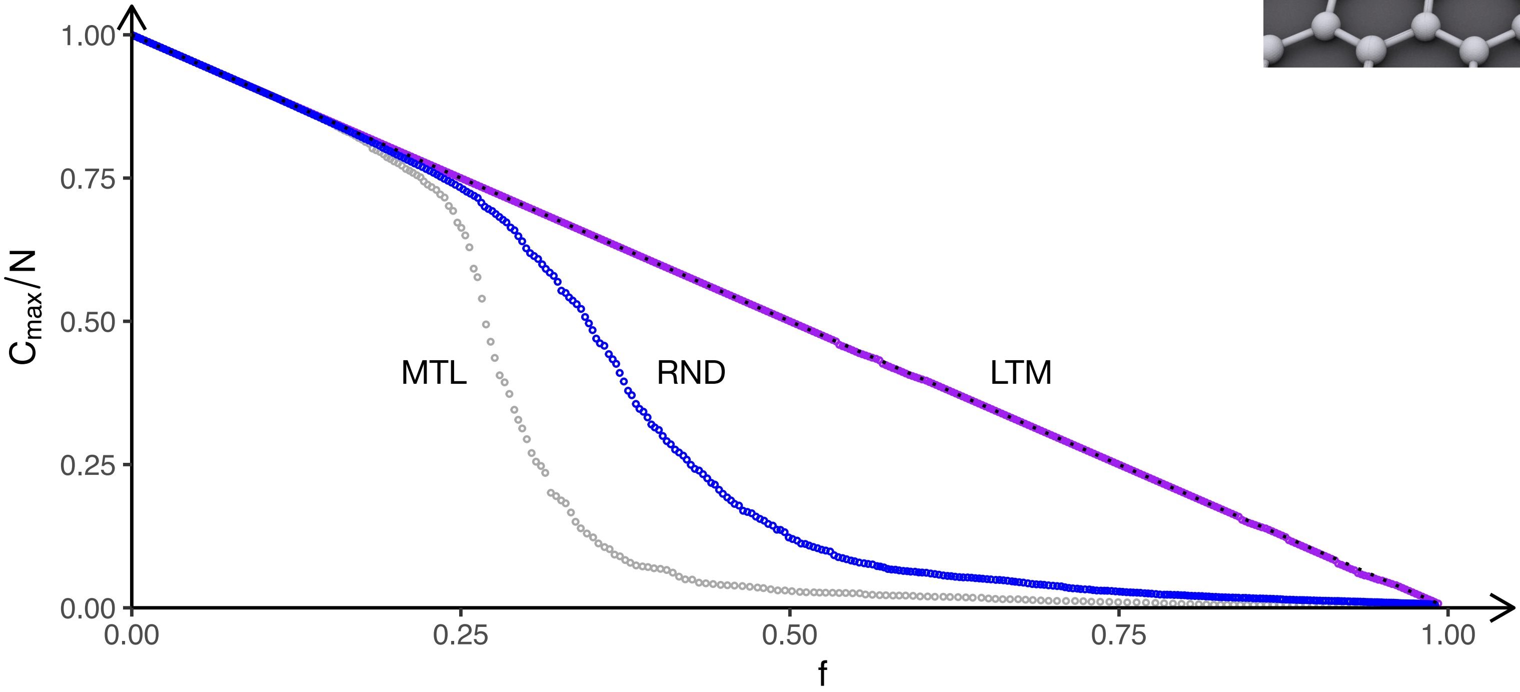
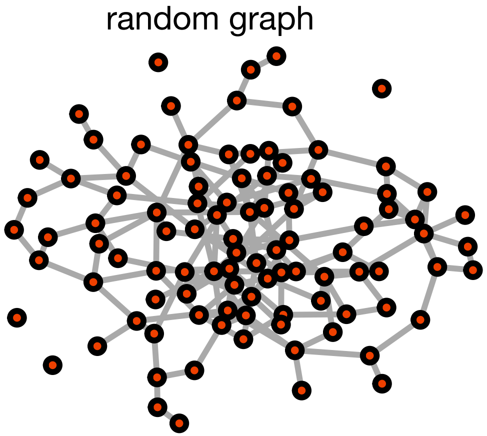
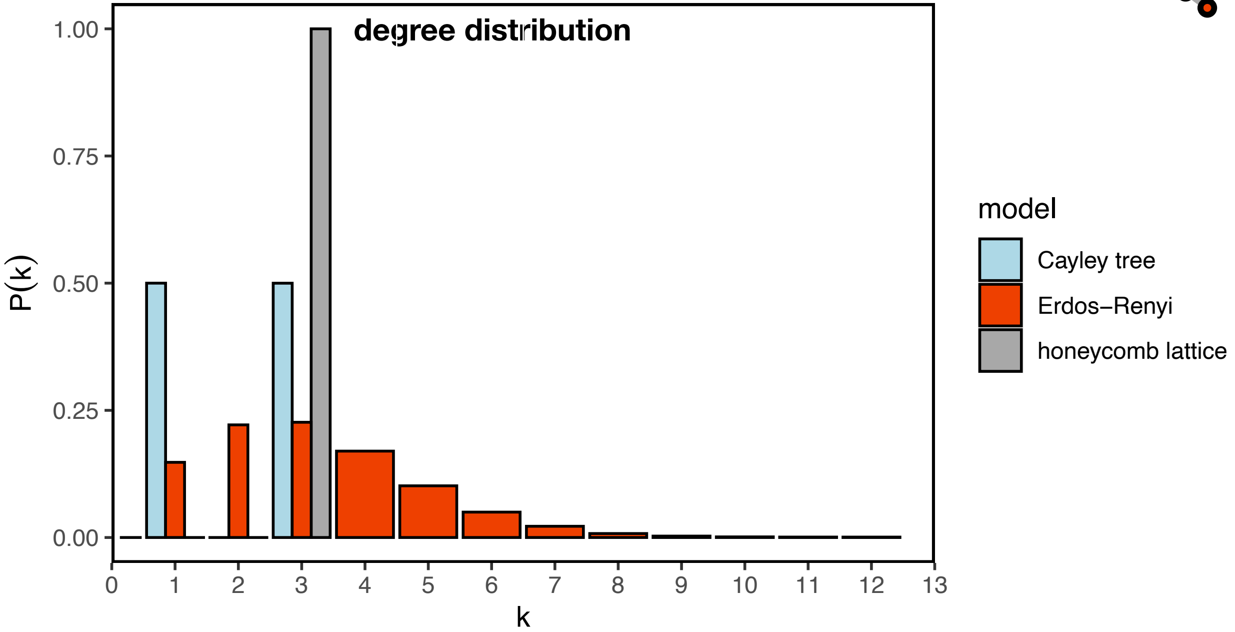


Erdos-Renyi graph

$$P(k) = \binom{N-1}{k} p^k (1-p)^{N-1-k}$$

binomial degree distribution





Erdos-Renyi graph

RND: random removal of nodes

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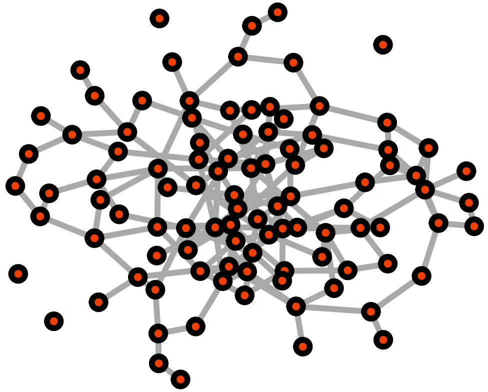
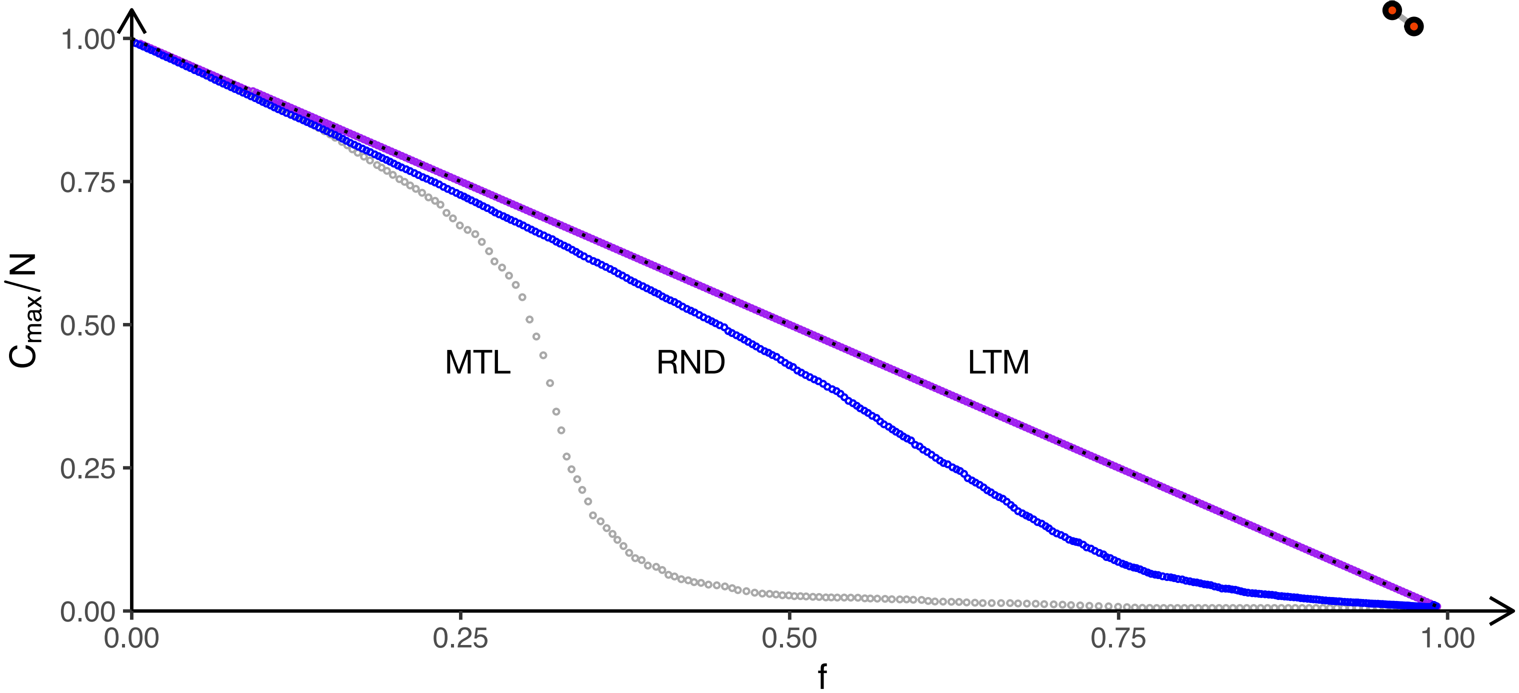
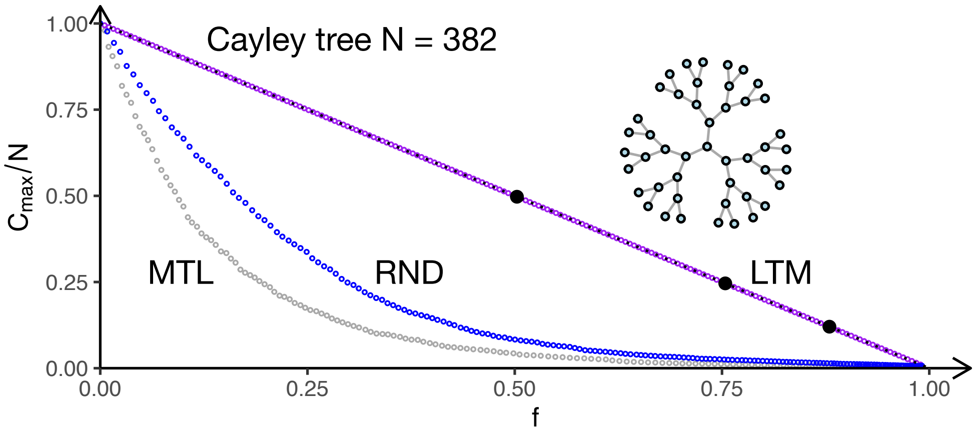
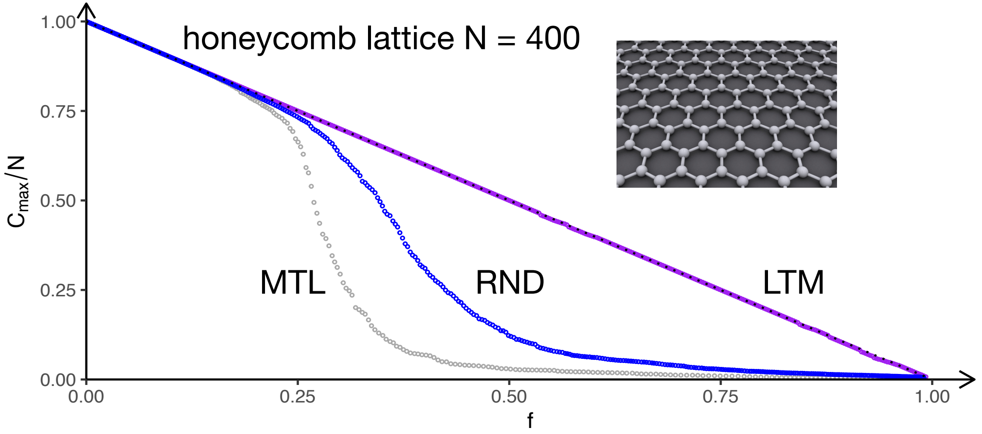
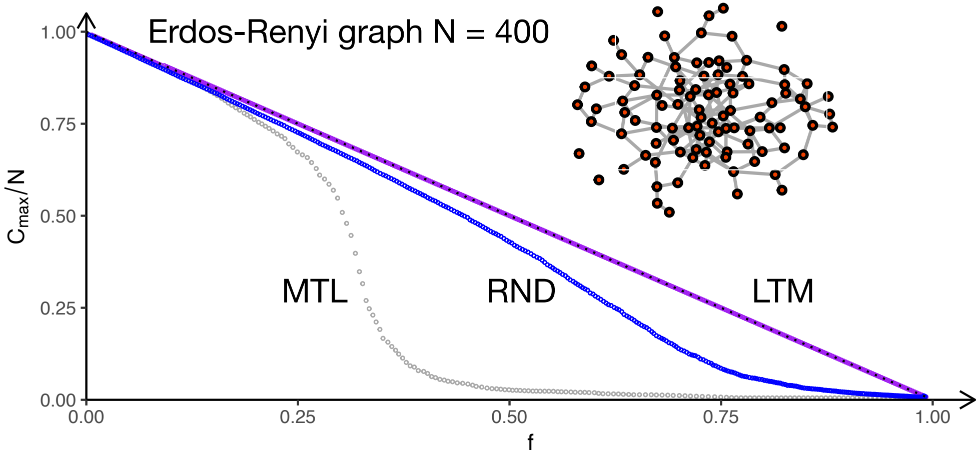


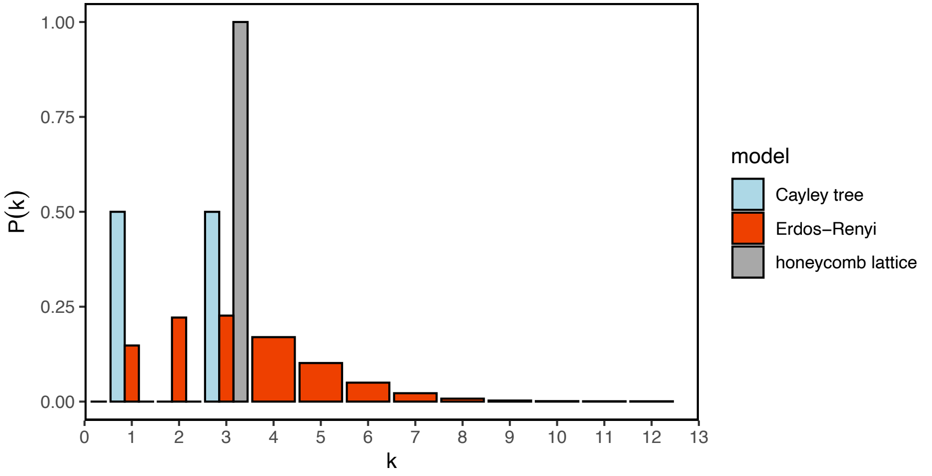


…discriminate

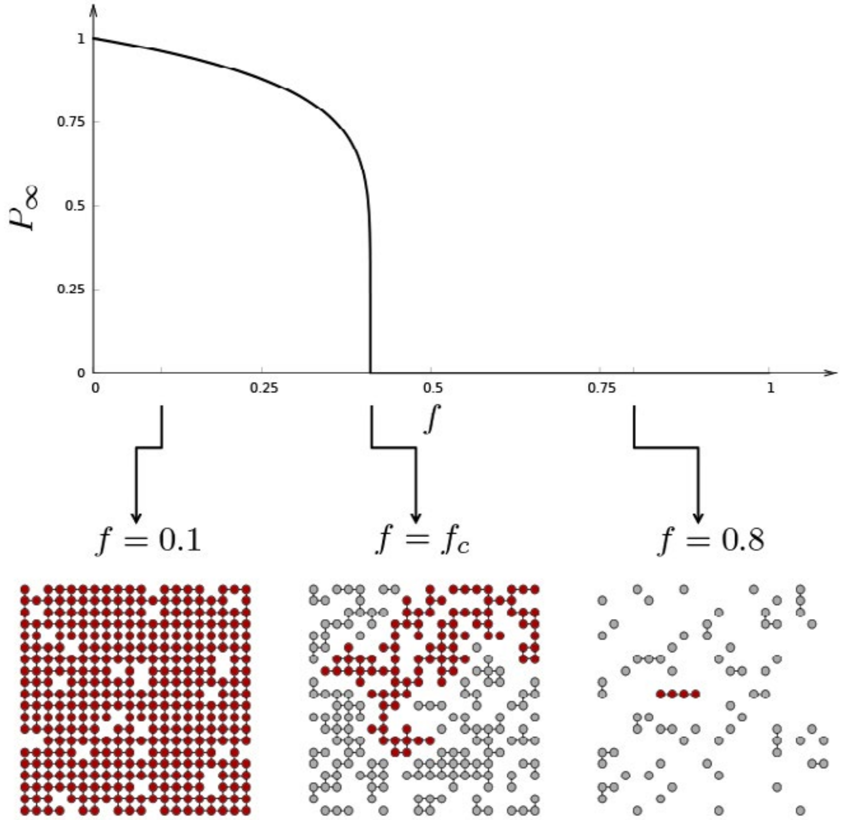








percolation theory



$0 < f < f_c :$ $f = f_c :$ $f > f_c :$

f = probability of a site to be empty

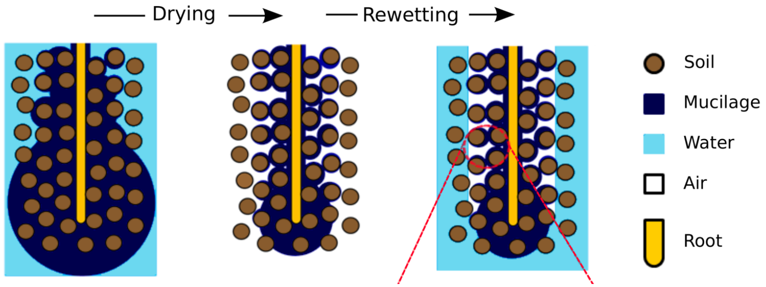
f = 0 => all sites in the lattice are occupied

f = 1 => all sites in the lattice are empty

P_∞ = probability that an occupied a site belongs to the giant cluster

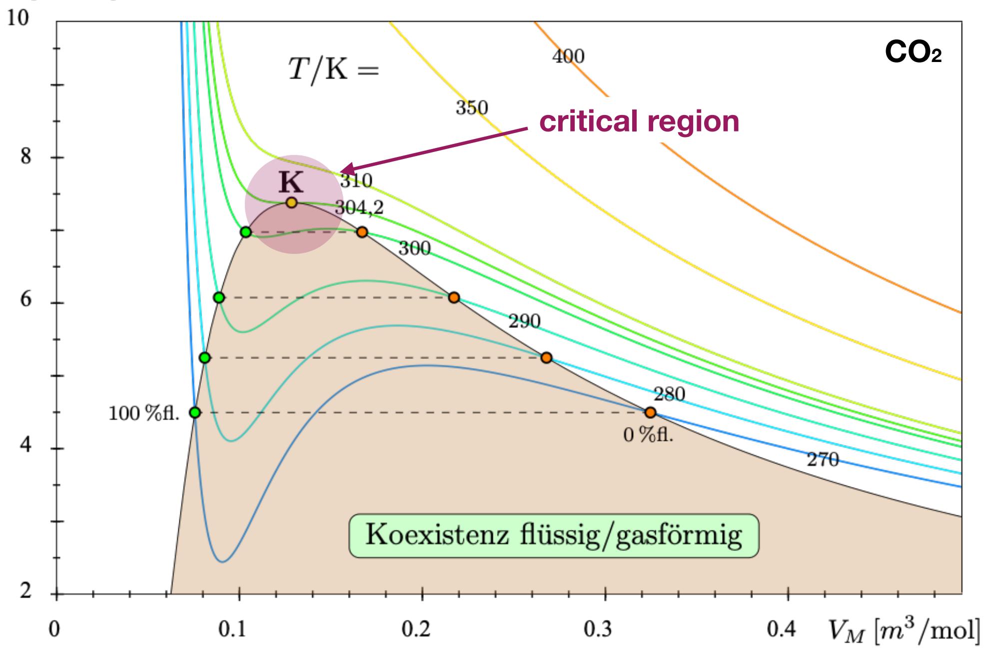
$P_\infty = \frac{C_{max}}{N}$ in the following

e.g. percolation applied to hydrology



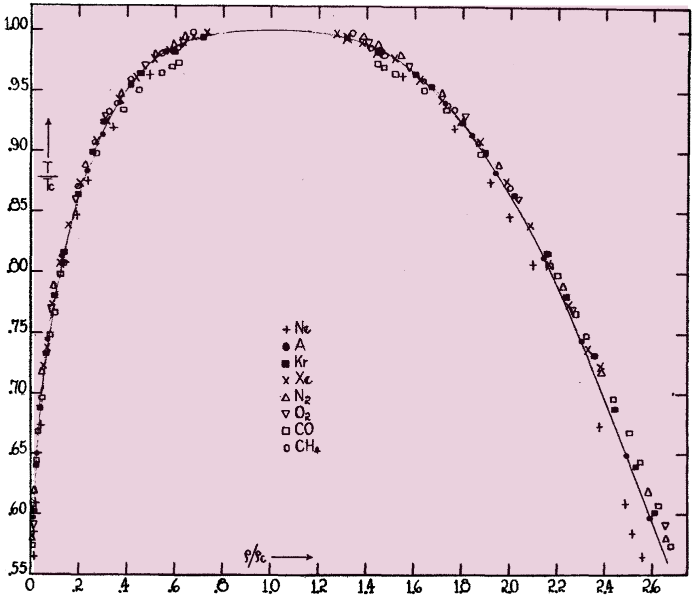
percolation is actually a critical phase transition

liquid-gas phase transition



Van der Waals equation 1873.

criticality $\frac{\rho_l - \rho_g}{\rho_c} \sim \left(\frac{T_c - T}{T_c}\right)^{\beta}$



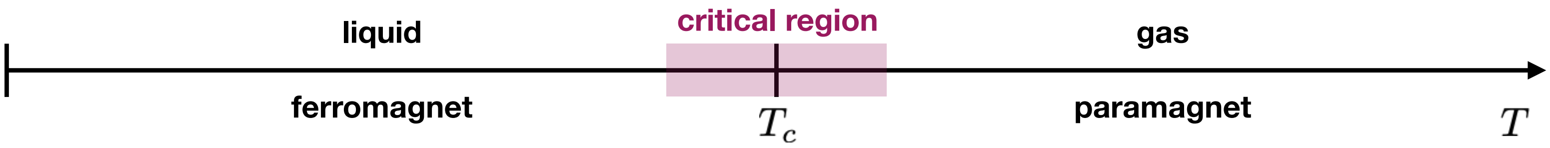
Guggenheim 1945.

universality: the fact that critical exponents (like beta) only depends on few details of the systems, such as the dimensionality, the range of interactions between particles, the symmetry of the problem.

Nobel prizes related to phase transitions

1910	Johannes Diderik van der Waals
1962	Lev Davidovich Landau
1968	Lars Onsager
1977	Philip Warren Anderson, Nevill Francis Mott, John Hasbrouck Van Vleck
1982	Kenneth G. Wilson
1991	Pierre-Gilles de Gennes
2001	Eric Allin Cornell, Carl Edwin Wieman, Wolfgang Ketterle
2016	David J. Thouless, John M. Kosterlitz, F. Duncan M. Haldane
2021	Giorgio Parisi, Klaus Hasselmann, Syukuro Manabe

critical phase transitions in physics



liquid

critical region

gas

ferromagnet

T_c

paramagnet

T

$$\frac{\rho_l - \rho_g}{\rho_c} \sim \left(\frac{T_c - T}{T_c}\right)^{\beta}$$

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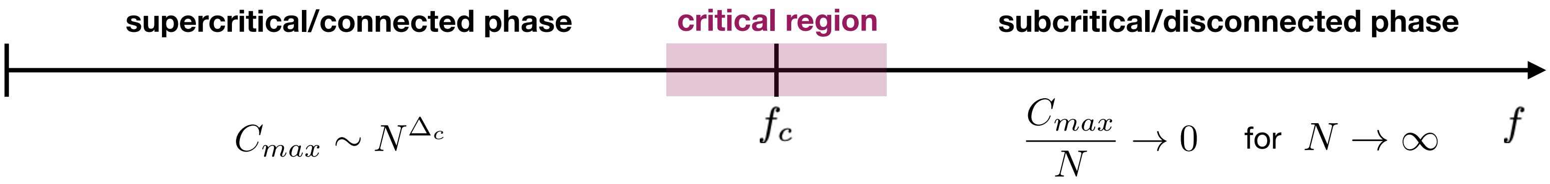
percolation phase transition in networks



supercritical/connected phase

critical region

subcritical/disconnected phase

$C_{max} \sim N^{\Delta_c}$

f_c

$\frac{C_{max}}{N} \to 0$ for $N \to \infty$

f

N = number of nodes in the network

$P_\infty = \frac{C_{max}}{N}$ is a good **order parameter**

$y = \frac{C_{max}}{N^{\Delta_c}}$ vs $x = \frac{f - f_c}{N^{\Delta_f}}$ **master curve**

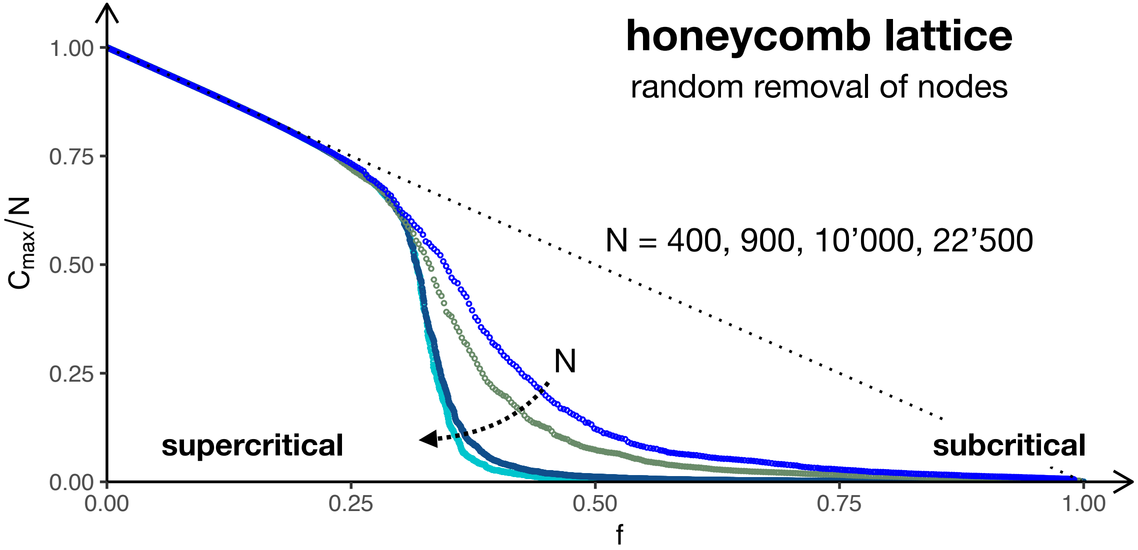
Δ_c and Δ_f depend on the **topology** of the network

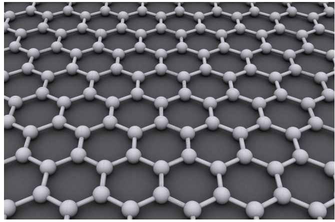
model	Δ_c	Δ_f
Erdos-Renyi	$2/3$	$1/3$
2D percolation	$303/288$	$3/8$

see e.g. Guimaraes, A.-L. (2020). Ann. Rev. Ecol. Evol. Syst. 51: 433-60

honeycomb lattice

random removal of nodes





increasing the number of nodes N the transition gets progressively sharper and approaches the **2D percolation** result

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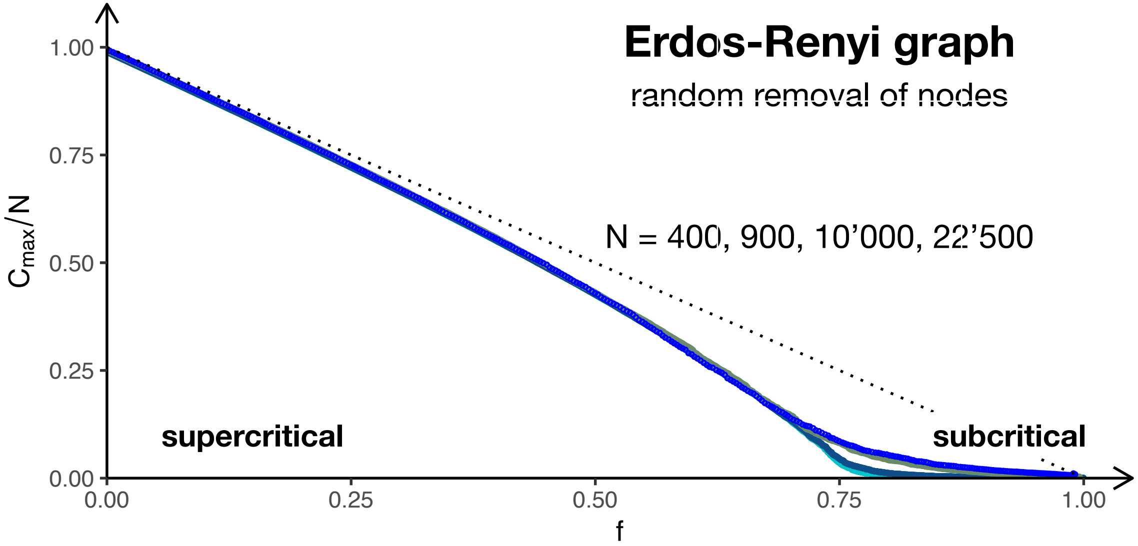
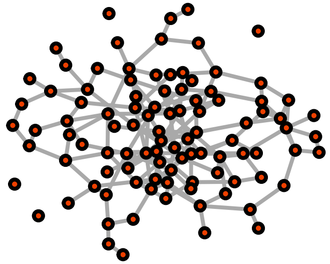
Erdos-Renyi graph

random removal of nodes





increasing the number of nodes N the transition gets progressively sharper as expected for the **Erdos-Renyi graph**

Molloy-Reed criterion

(1995) M. Molloy, B. Reed, Random Struct. Algorithms 6, 161

Molloy-Reed criterion

valid for every random network

supercritical/connected phase

$C_{max} \sim N^{\Delta_c}$

$\boxed{\kappa \geq 2}$

f_c

subcritical/disconnected phase

$\frac{C_{max}}{N} \to 0$ for $N \to \infty$

$\boxed{\kappa < 2}$

f

$$\kappa = \sum_i k_i P(k_i | i \leftrightarrow j)$$

$P(A|B)$ **conditional probability**

A: i-th node has degree k

B: i-th node is connected to the j-th node

generally $\kappa(f)$ therefore we can define the critical threshold as $\boxed{\kappa(f_c) = 2}$

in terms of the original distribution $\boxed{f_c = 1 - \frac{1}{\kappa - 1}}$

Molloy-Reed criterion

valid for every random network

A : i has degree k_i

B : i connected to j

Bayes thm

$$P(A|B) = \frac{P(B|A)\, P(A)}{P(B)} = \frac{k_i}{N-1}\,\frac{N-1}{\langle k \rangle}\, p(k_i)$$

Adjacency Mat.

	1	2	3	4
1	X			1
2		X		
3			X	
4	1			X

$$P(i \longleftrightarrow j) = P(B) = \frac{2\overset{\#\,\text{links}}{L}}{N(N-1)} = \frac{\langle k \rangle}{N-1}$$

ACHTUNG

It works ONLY if a priori i can be connected with same prob. to ANY other node!

$$P(i \leftrightarrow j \,|\, k_i) = P(B|A) = \frac{k_i}{N-1}$$

k_i	1	2	3	...
$P(B\|A)$	$\frac{1}{N-1}$	$\frac{2}{N-1}$	$\frac{3}{N-1}$	..

Molloy-Reed criterion

valid for every random network

$$\text{Kappa} = \sum_i k_i \frac{k_i}{\cancel{N-1}} \frac{\cancel{N-1}}{\langle k \rangle} p(k_i) = \frac{\langle k^2 \rangle}{\langle k \rangle} \geq 2$$

in order for the Network to be connected.

$$\text{Kappa} = \frac{\langle k^2 \rangle}{\langle k \rangle} \geq 2 \Rightarrow \text{Supercritical phase}$$

$$C_{MAX} \sim N^{\Delta_c}$$

The behavior of $\langle k \rangle$ & $\langle k^2 \rangle$ as $N \to \infty$: crucial to assess the ROBUSTNESS of a random network (e.g. ER graph, scale free)

Molloy-Reed criterion

valid for every random network

Erdos-Renyi graph

$$P(k) = \binom{N-1}{k} p^k (1-p)^{N-1-k}$$

binomial degree distribution

$$\langle k^2 \rangle - \langle k \rangle^2 = \langle k \rangle \quad \Rightarrow \quad \langle k^2 \rangle = \langle k \rangle^2 + \langle k \rangle$$

property binomial distribution

$$\kappa = \frac{\langle k^2 \rangle}{\langle k \rangle} = \langle k \rangle + 1 \geq 2 \quad \Rightarrow \quad \boxed{\langle k \rangle \geq 1}$$

known result

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scale-free network

$$P(k) = Ck^{-\gamma}$$

C generally depends on k_{min} and $k_{max} \sim N^{1/(\gamma-1)}$

in the limit $N \to \infty$

$$\boxed{\begin{array}{ll} f_c \to 1 & \text{for } 2 < \gamma \leq 3 \\ f_c < 1 & \text{for } \gamma > 3 \end{array}}$$

Molloy-Reed criterion

valid for every random network

$$\int_{k_{min}}^{k_{max}} k^{\alpha}\, dk = \frac{1}{\alpha+1}\left[k^{\alpha+1} \right]_{k_{min}}^{k_{max}}$$

$$\langle k \rangle = C \int_{k_{min}}^{k_{max}} k\, k^{-\gamma}\, dk = C \int_{k_{min}}^{k_{max}} k^{1-\gamma}\, dk \approx \frac{C}{2-\gamma}\, k_{max}^{2-\gamma}$$

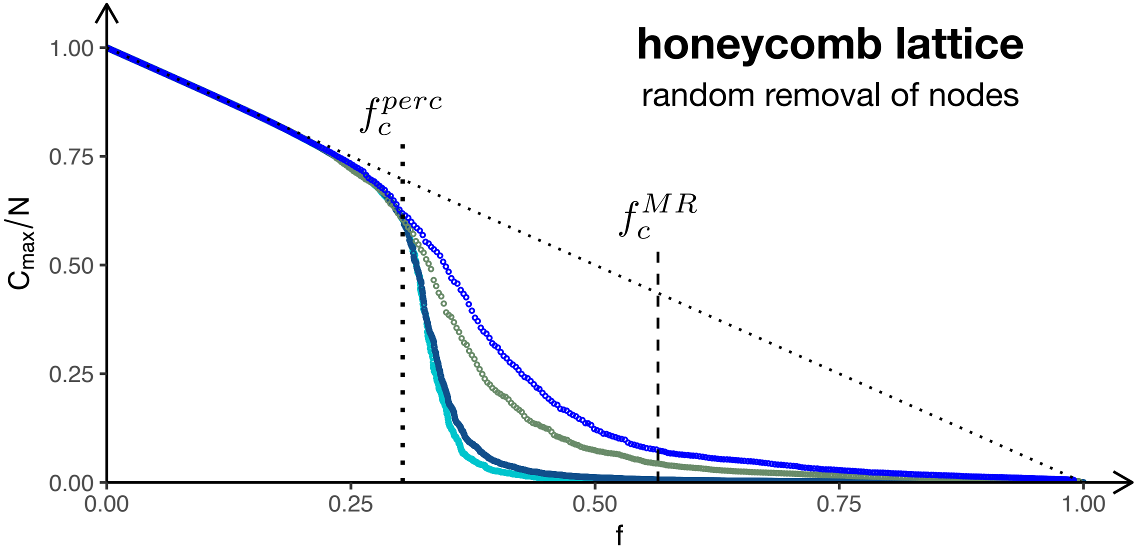
$$\langle k^2 \rangle = C \int_{k_{min}}^{k_{max}} k^2\, k^{-\gamma}\, dk = C \cdot\cdot \int_{k_{min}}^{k_{max}} k^{2-\gamma}\, dk \approx \frac{C}{3-\gamma}\, k_{max}^{3-\gamma}$$

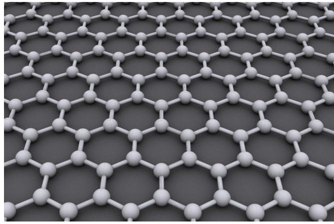
for $N \to \infty$

$\langle k \rangle < \infty$ if $\gamma > 2$

Genuine scale free: for $N \to \infty$ & RND removal, always in super cr. ph.

$\langle k^2 \rangle < \infty$ if $\gamma > 3$ $\Longrightarrow$ $\langle k^2 \rangle \to \infty$ $\boxed{2 < \gamma \leq 3}$





the **Molloy-Reed** criterion **fails** to predict the correct critical fraction; **2D percolation** is instead accurate

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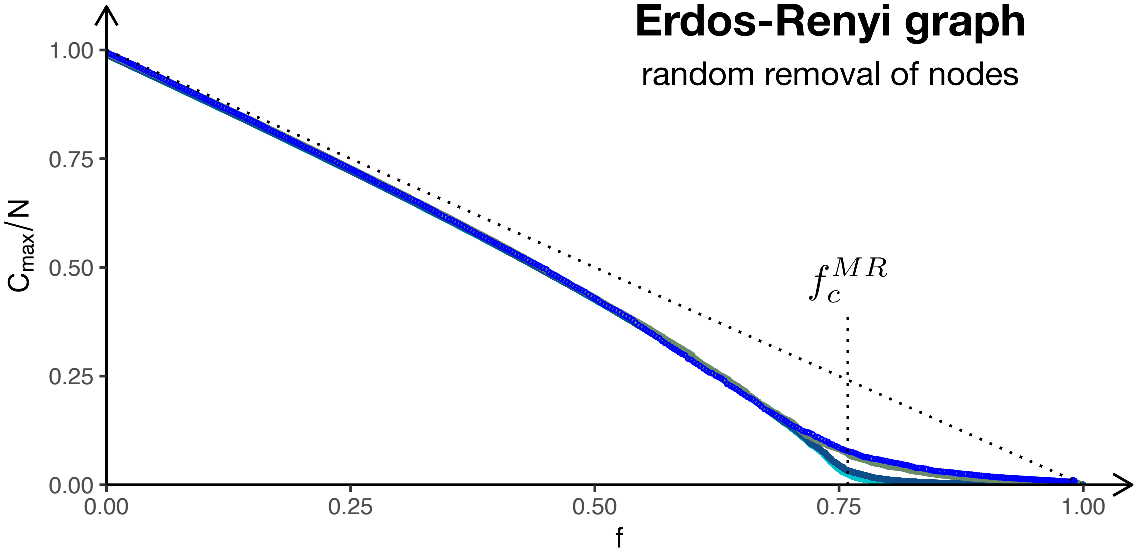
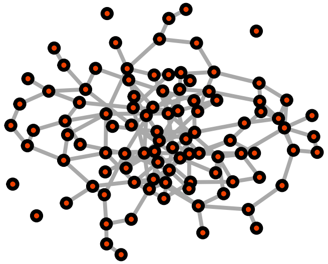
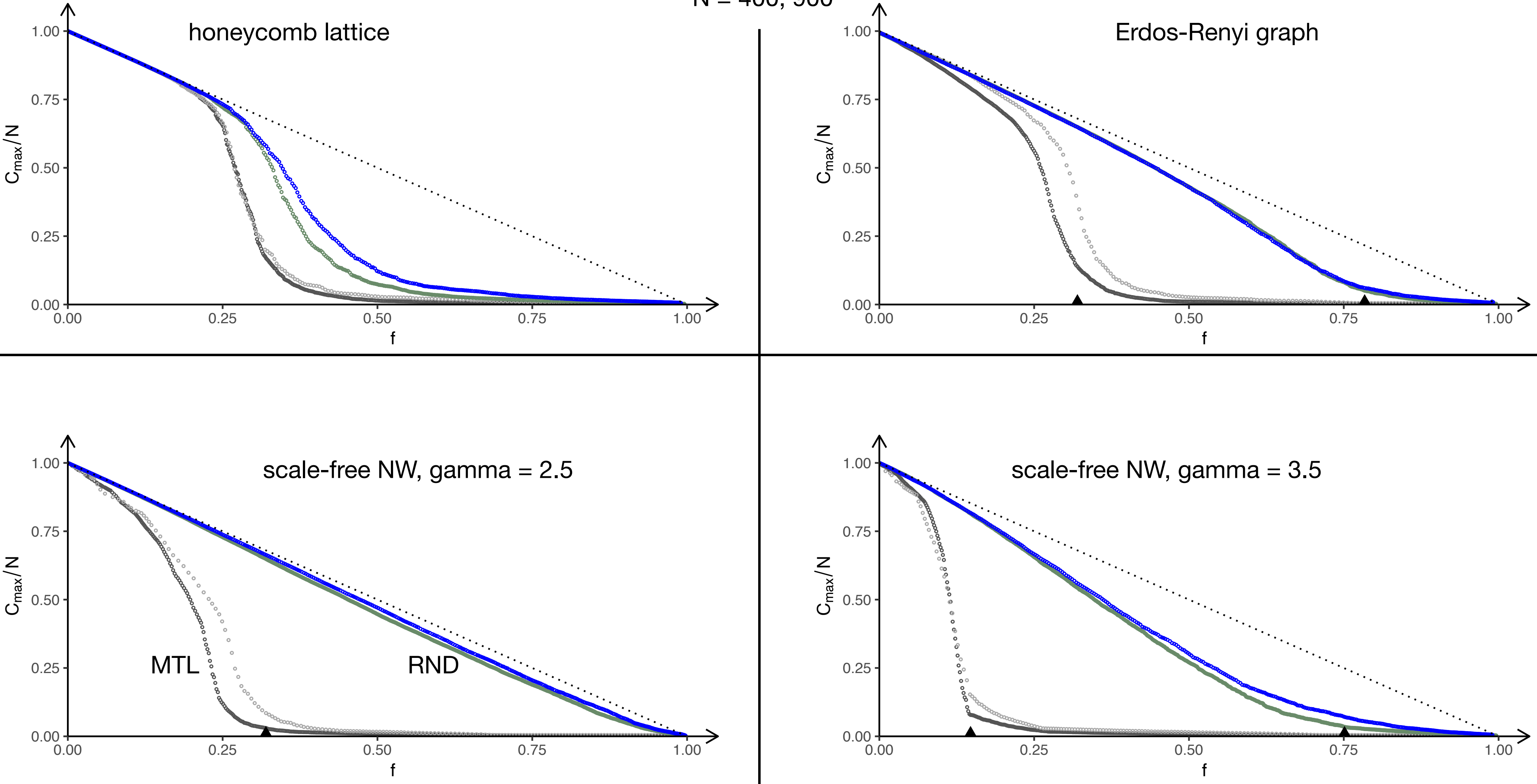




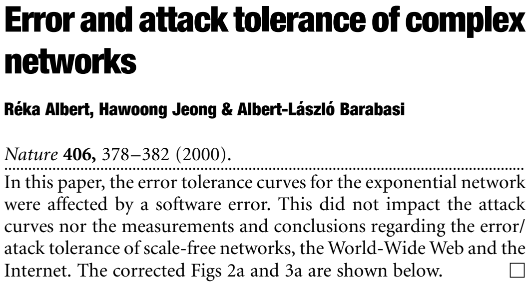
the **Molloy-Reed** criterion predicts the correct critical fraction

Molloy-Reed critical threshold in different models

N = 400, 900



suggested readings



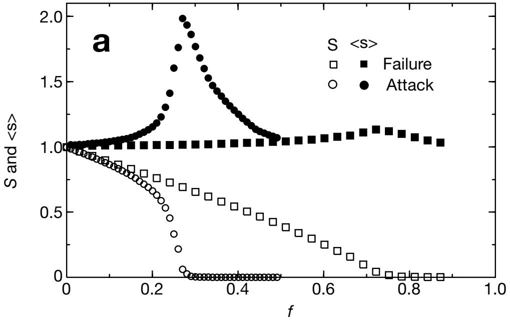
Error and attack tolerance of complex networks

Réka Albert, Hawoong Jeong & Albert-László Barabasi

Nature **406,** 378–382 (2000).

In this paper, the error tolerance curves for the exponential network were affected by a software error. This did not impact the attack curves nor the measurements and conclusions regarding the error/atack tolerance of scale-free networks, the World-Wide Web and the Internet. The corrected Figs 2a and 3a are shown below. ☐

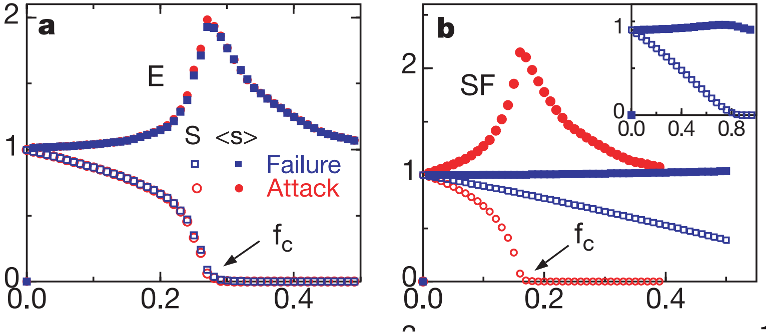
correct one

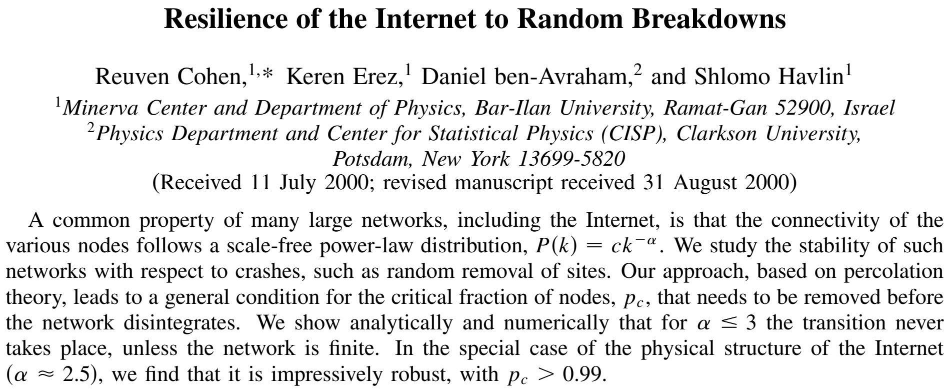


Error and attack tolerance of complex networks

Réka Albert, Hawoong Jeong & Albert-László Barabási

Department of Physics, 225 Nieuwland Science Hall, University of Notre Dame, Notre Dame, Indiana 46556, USA





Resilience of the Internet to Random Breakdowns

Reuven Cohen,[1,*] Keren Erez,[1] Daniel ben-Avraham,[2] and Shlomo Havlin[1]

[1]*Minerva Center and Department of Physics, Bar-Ilan University, Ramat-Gan 52900, Israel*
[2]*Physics Department and Center for Statistical Physics (CISP), Clarkson University,*
Potsdam, New York 13699-5820

(Received 11 July 2000; revised manuscript received 31 August 2000)

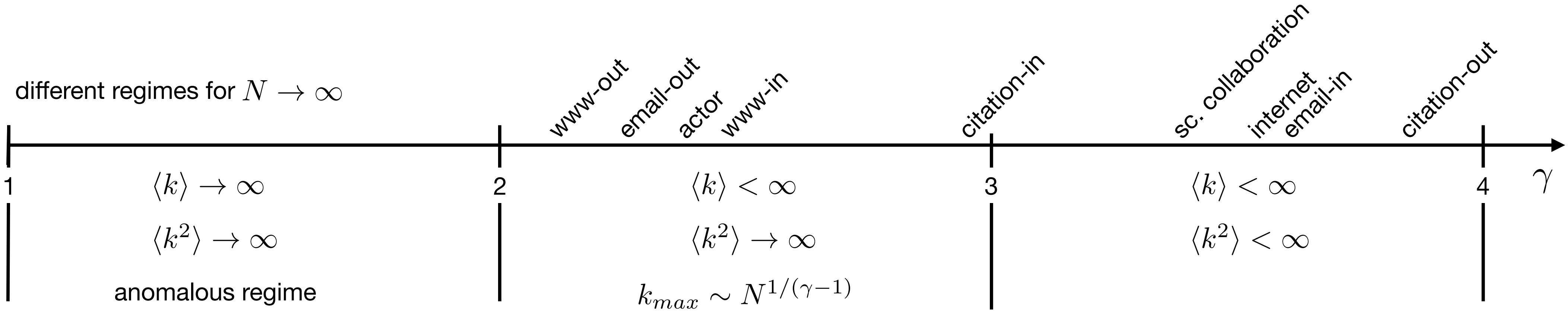
A common property of many large networks, including the Internet, is that the connectivity of the various nodes follows a scale-free power-law distribution, $P(k) = ck^{-\alpha}$. We study the stability of such networks with respect to crashes, such as random removal of sites. Our approach, based on percolation theory, leads to a general condition for the critical fraction of nodes, p_c, that needs to be removed before the network disintegrates. We show analytically and numerically that for $\alpha \leq 3$ the transition never takes place, unless the network is finite. In the special case of the physical structure of the Internet ($\alpha \approx 2.5$), we find that it is impressively robust, with $p_c > 0.99$.

robustness of real networks

$$\boxed{f_c = 1 - \frac{1}{\kappa - 1}}$$

works for RND removal

network	N	L	$\langle k \rangle$	$\langle k^2 \rangle$	γ	κ
www-in	325'729	1'497'134	4.6	1'546	2	336
www-out	325'729	1'497'134	4.6	482	2.31	105
email-in	57'194	103'521	1.81	1'546	3.43	19
email-out	57'194	103'521	1.81	482	2.03	643
citation-in	449'673	4'689'479	10.43	971.5	3	93
citation-out	449'673	4'689'479	10.43	198.8	4	19
actor	702'388	29'397'908	83.71	47'353	2.12	565
sc. collaboration	23'133	93'439	8.08	178.2	3.35	22
Internet	192'244	609'066	6.34	240.1	3.42	38
power grid	4'941	6'594	2.67	10.3	[Exp.]	3.86

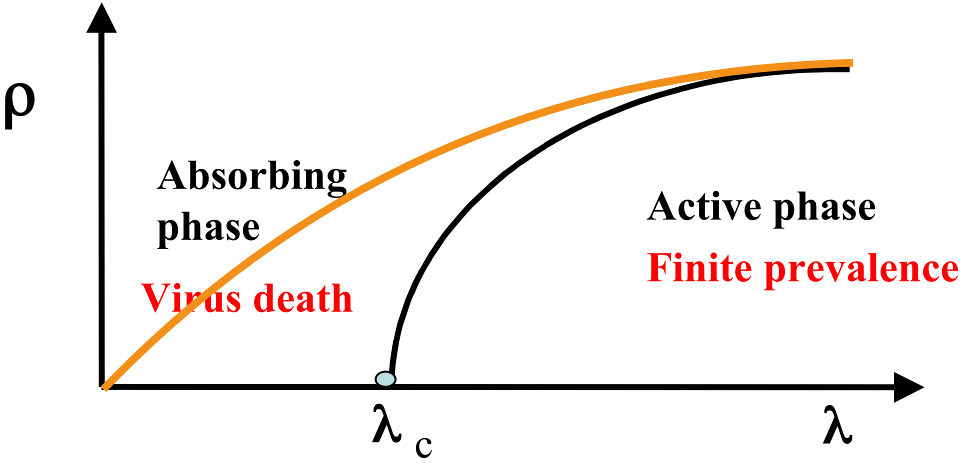


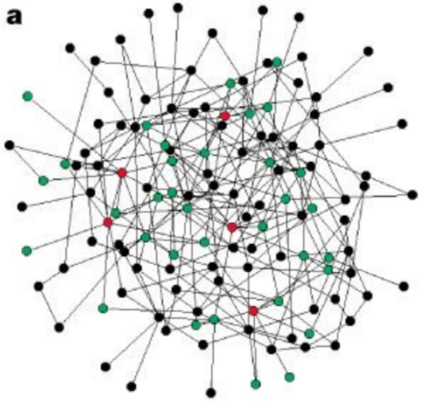
different regimes for $N \to \infty$

1–2	2–3	3–4
$\langle k \rangle \to \infty$	$\langle k \rangle < \infty$	$\langle k \rangle < \infty$
$\langle k^2 \rangle \to \infty$	$\langle k^2 \rangle \to \infty$	$\langle k^2 \rangle < \infty$
anomalous regime	$k_{max} \sim N^{1/(\gamma-1)}$	

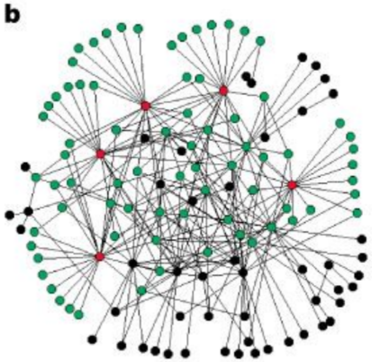
Taken from Barabási, A.-L. (2016). *Network Science*. Cambridge University Press.

how is this related to epidemic threshold of viruses?





Exponential



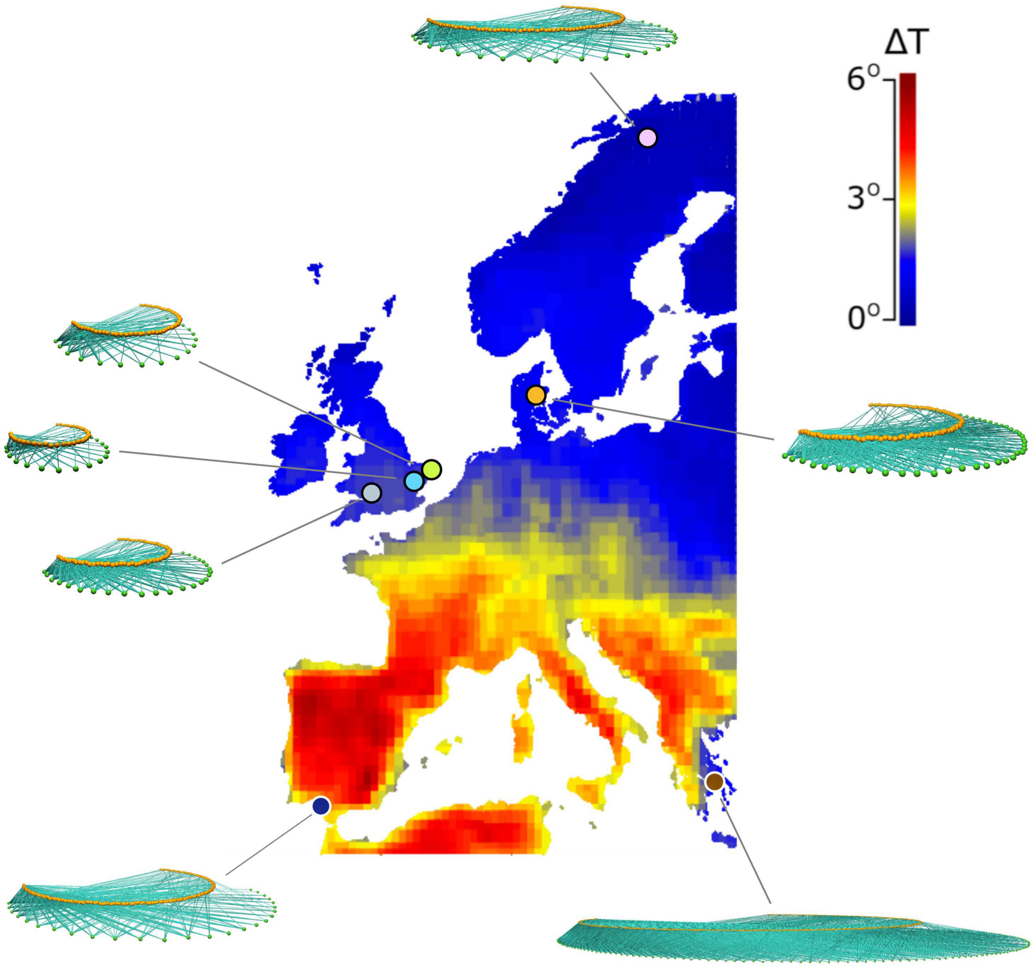
Scale-free

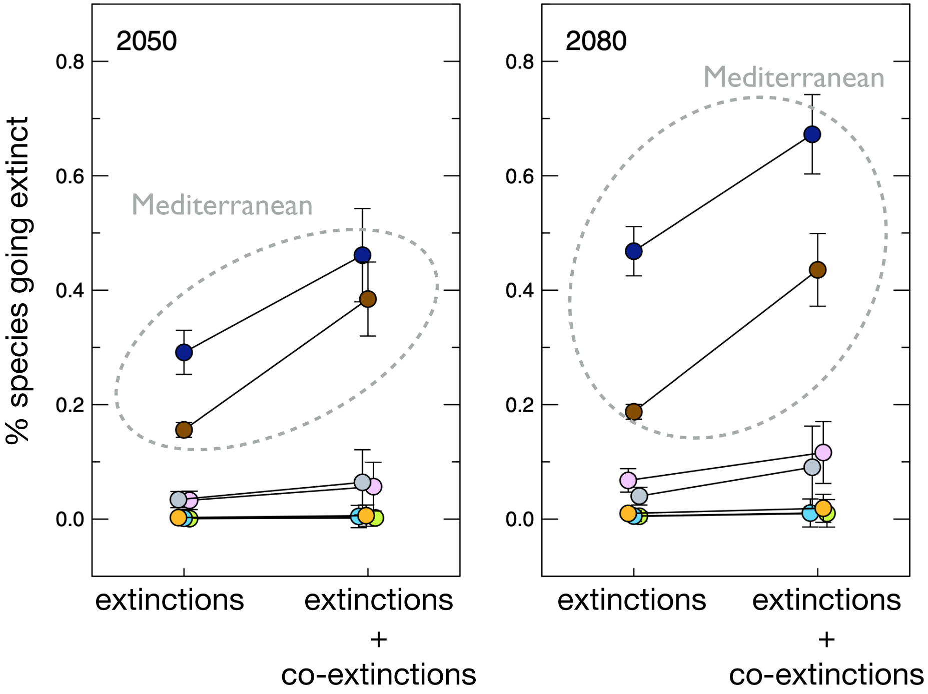
Pastor Satorras and Vespignani (2001)

what is relevant for ecology…
…according to a physicist

We can possibly discuss about what follows next week during the general discussion

ecologically-driven removal of nodes





Bascompte, García, Ortega, Rezende, and Pironon (2019). *Sci. Adv.* 5: eaav2539

can we learn something about habitat restoration from physics?

Received: 14 October 2020 | Accepted: 16 February 2021

DOI: 10.1111/1365-2656.13450

RESEARCH ARTICLE

Journal of Animal Ecology

Habitat restoration in spatially explicit metacommunity models

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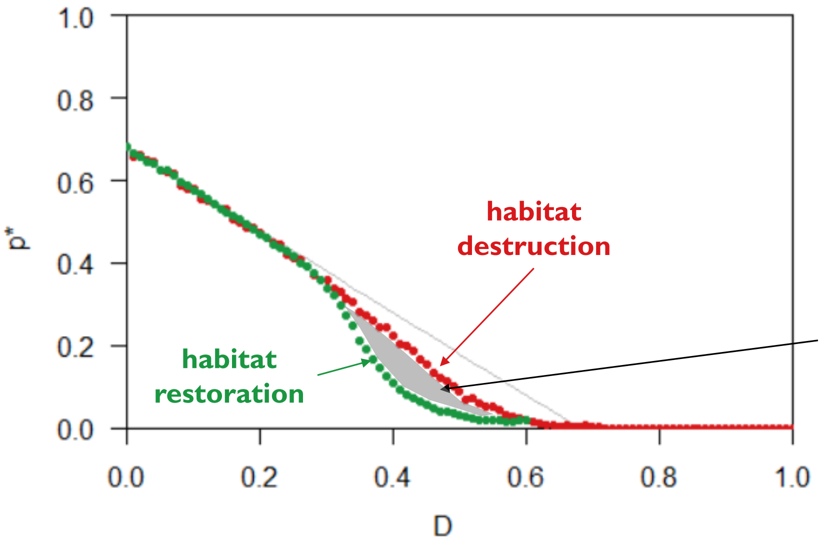
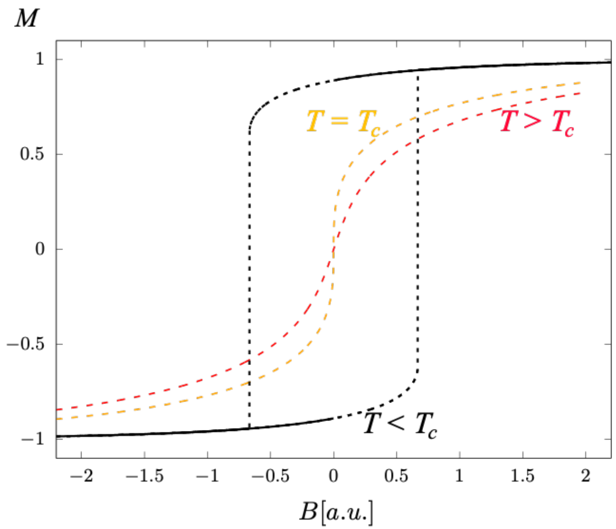


habitat destruction

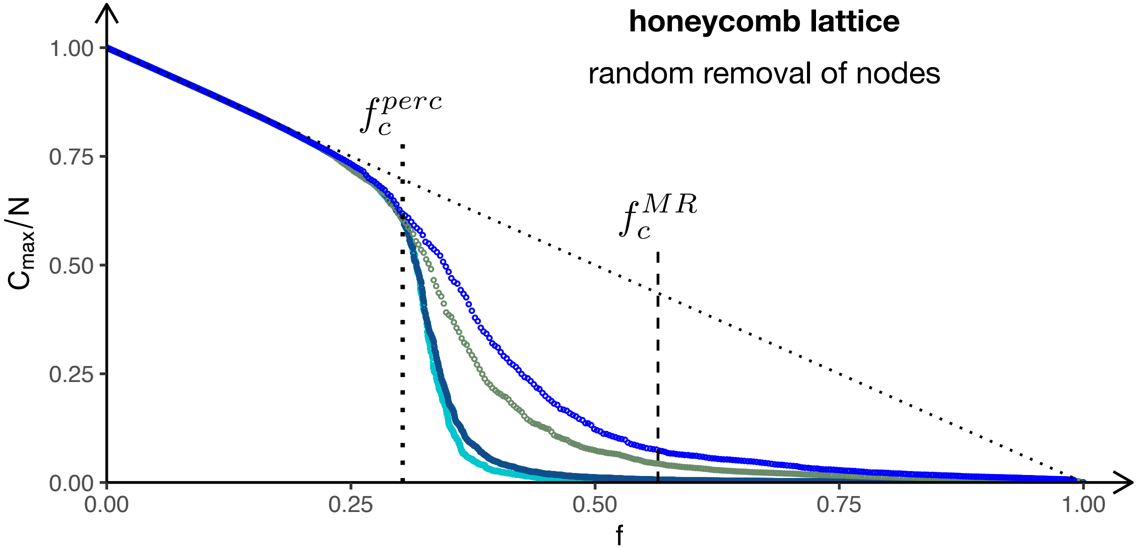
habitat restoration

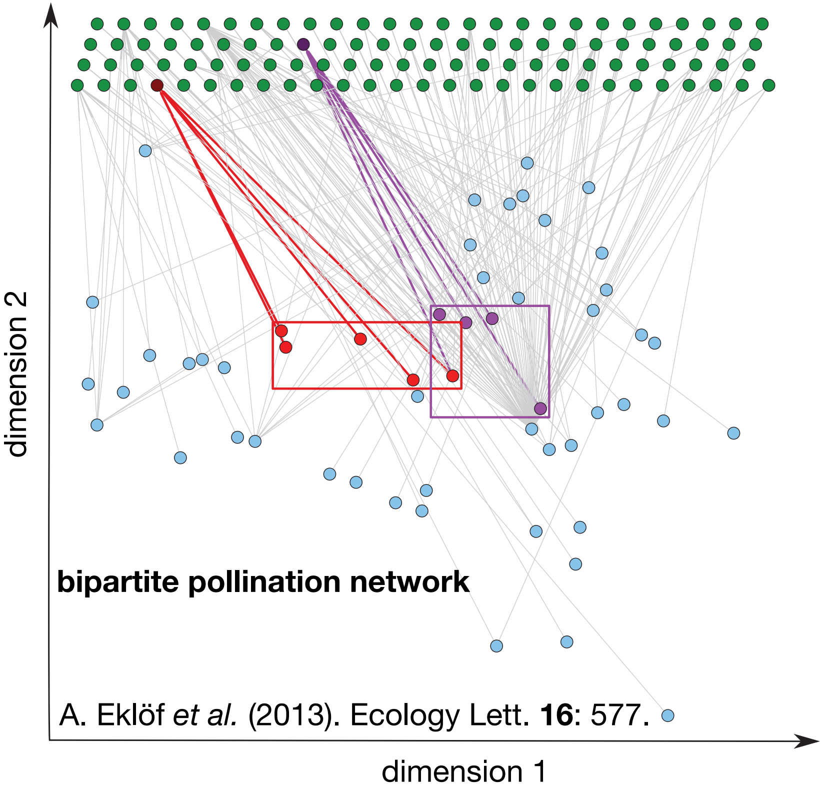
restoration lag

magnetic hysteresis appears in magnets in the **supercritical** (ferromagnetic) phase



1. are ecological networks really random?





A. Eklöf *et al.* (2013). Ecology Lett. **16**: 577.

2. what role play its finite size or the sampling efforts on assessing the robustness of an ecological network?

Thank you!

LEGENDA

N = # of nodes in a network (initial in simulations))

n = current # of nodes during removal simulation

$f = 1 - \frac{n}{N}$ fraction removed nodes

C_{MAX} = size of the largest cluster

f_c = critical fraction removed nodes

$C_{MAX} \sim N^{\Delta_c}$ Δ_c exponent

$N^{\Delta f}$ Δ_f exponent

N^{Δ_f} "sharpness" window of critical region

$x = \frac{f - f_c}{N^{\Delta f}}$ horizontal variable of the master curve

$y = \frac{C_{max}}{N^{\Delta_c}}$ vertical variable of the master curve

$K = [\text{Kappa}] = \frac{\langle k^2 \rangle}{\langle k \rangle}$

$\langle k \rangle$ = average degree of the network

$\langle k^2 \rangle$ = average squared degree of the network

N.B. both $\langle k \rangle$ & $\langle k^2 \rangle$ can be computed with the initial degree distribution as in $f_c = 1 - \frac{1}{\text{Kappa} - 1}$

OR with the current degree distribution

as in Simulation and exercise session (e.g., df-hc-100)