# network robustness

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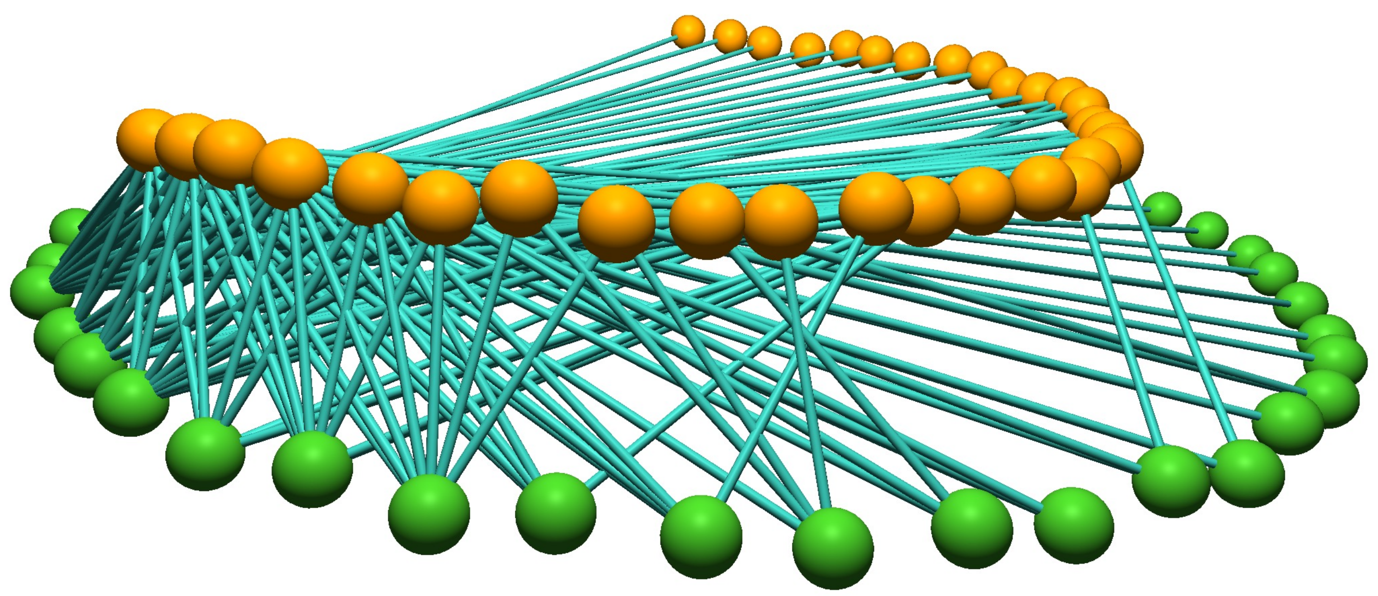
# why should we care about network robustness?

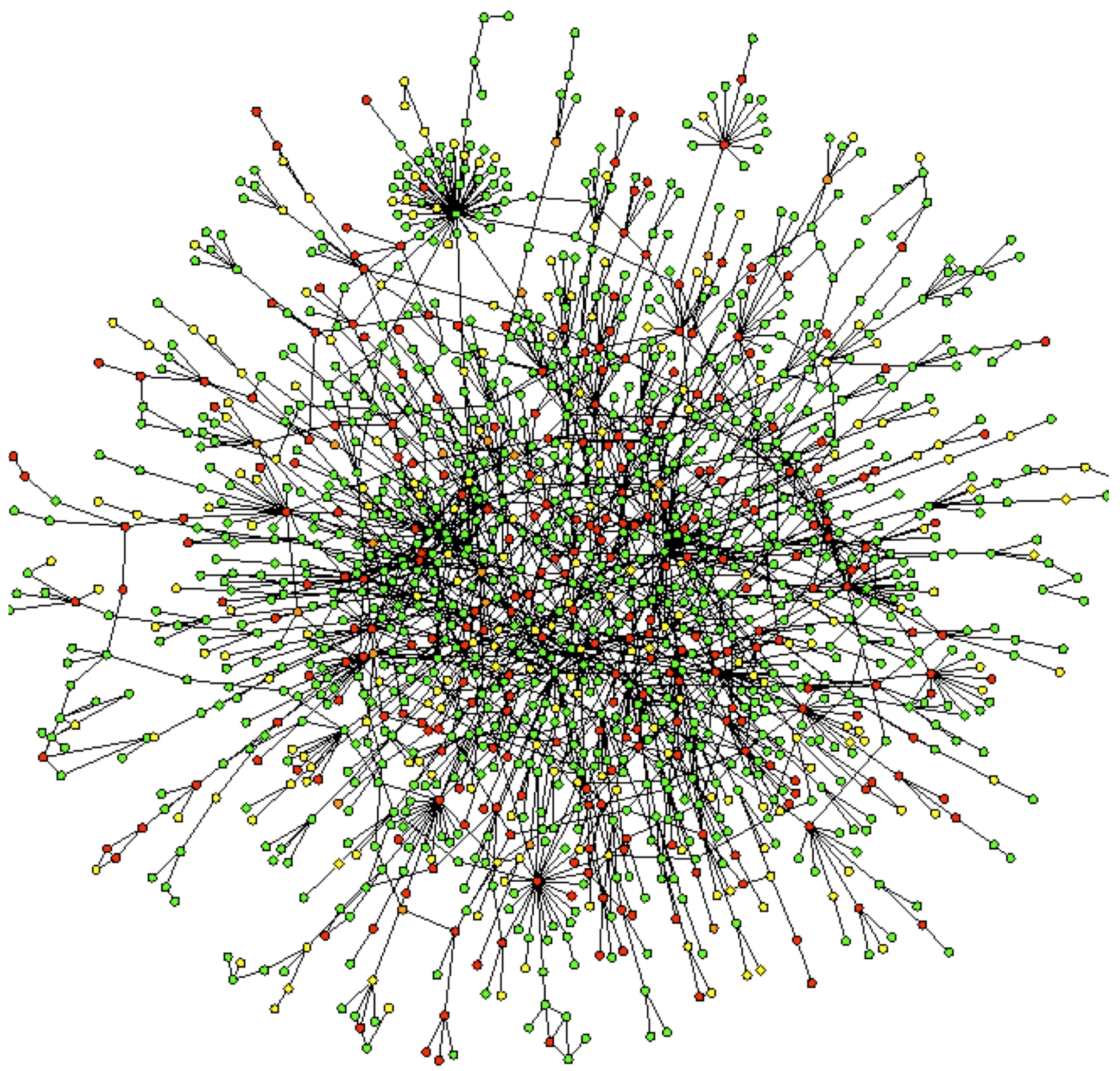
ecological networks



1. network robustness and percolation transition

criminal network

sexual contacts



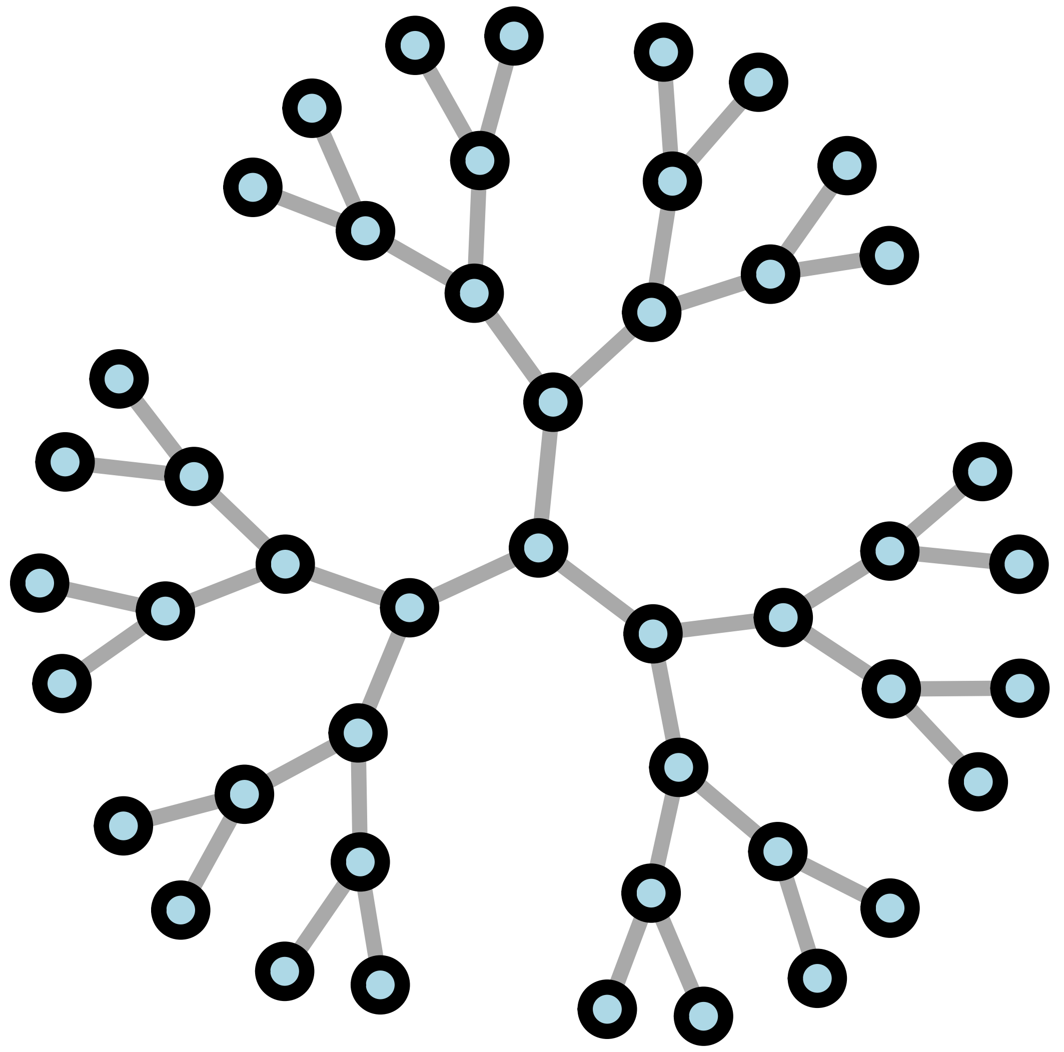
2. Molloy-Reed criterion

3. what is relevant for ecology according to a physicist

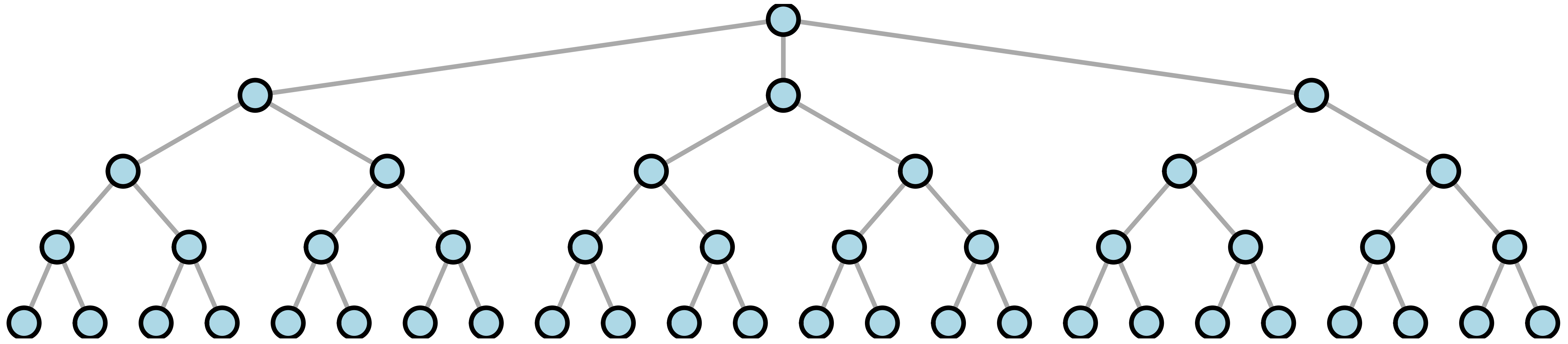
# network robustness
# and
# percolation transition

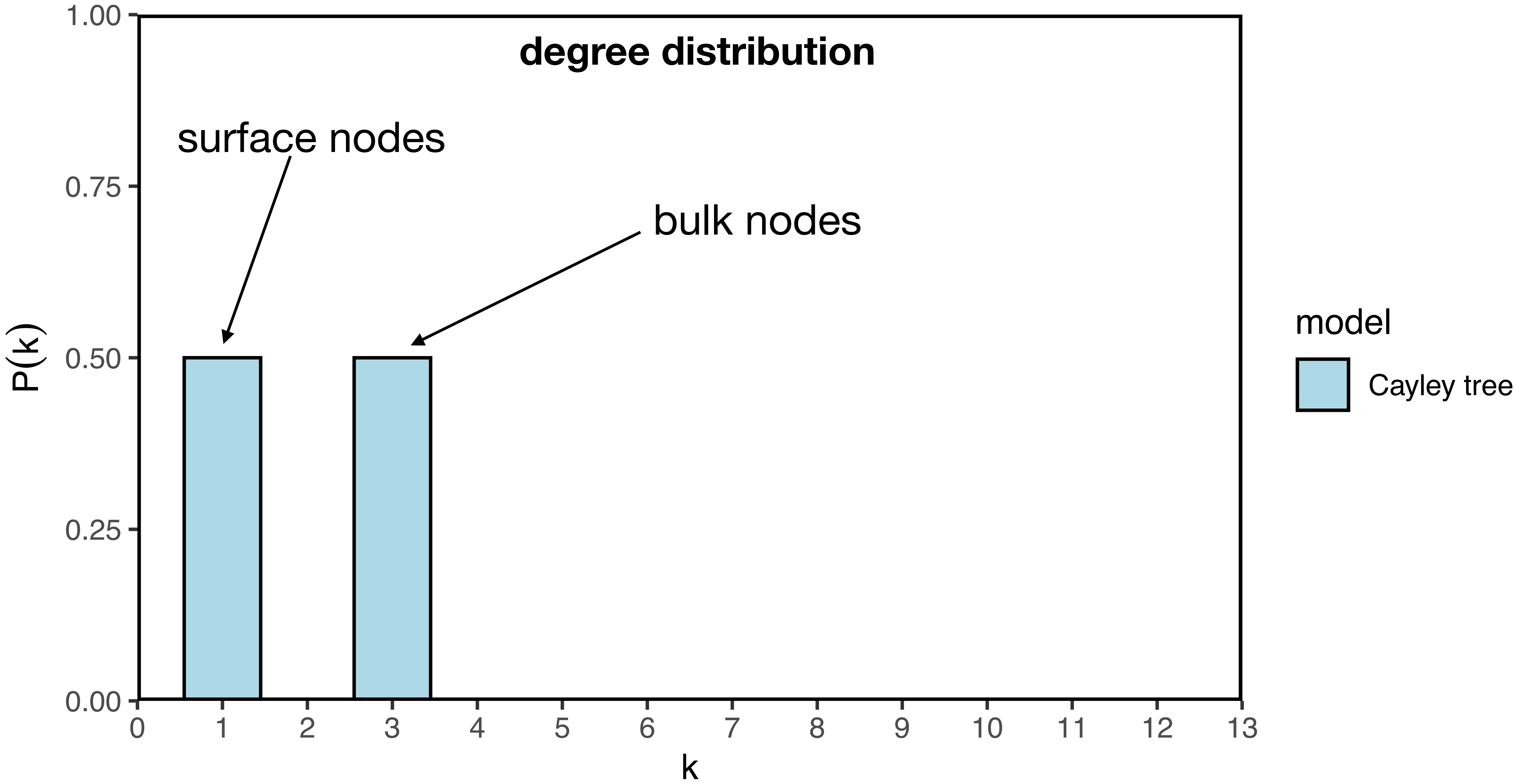
# Cayley tree with k=3

star-like representation



tree-like representation





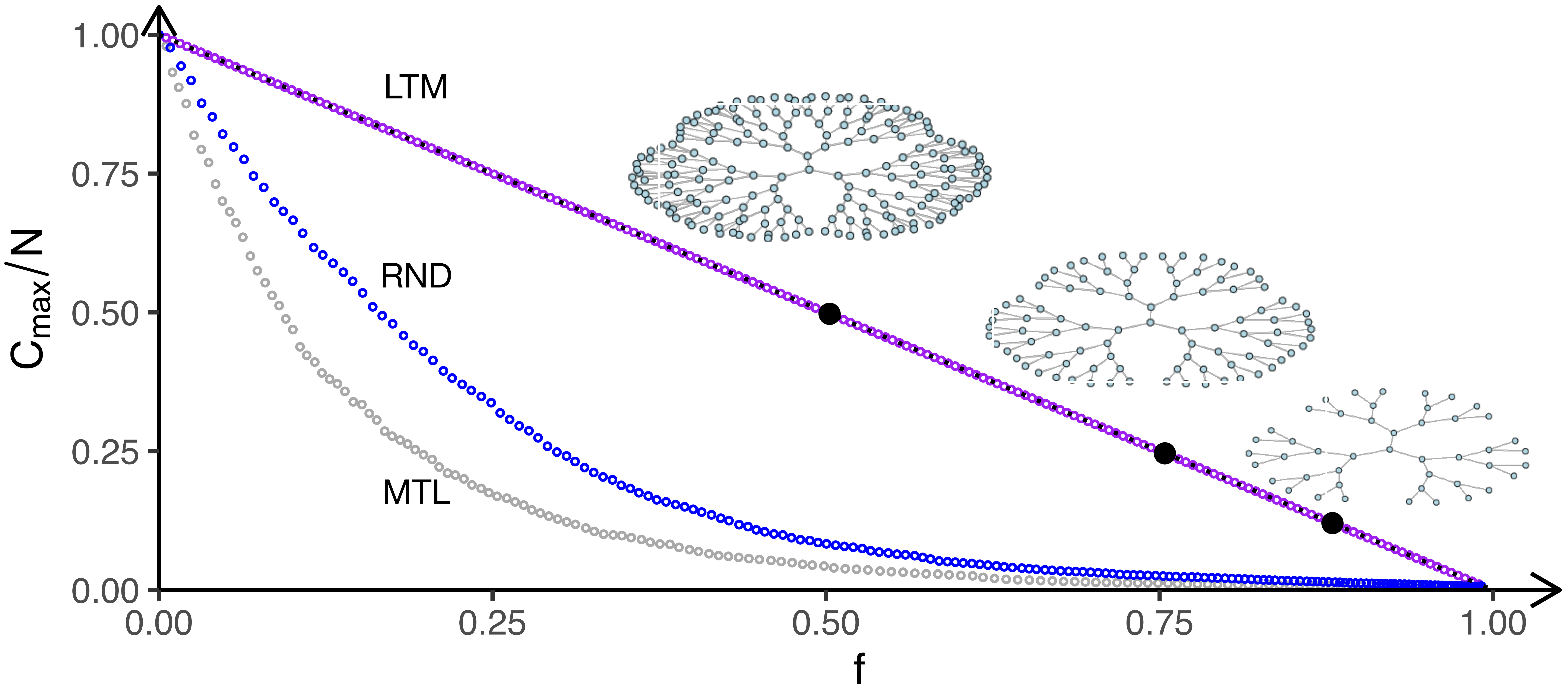

k = number of links per node

# Cayley tree

RND: random removal of nodes

MTL: from most connected to least connected

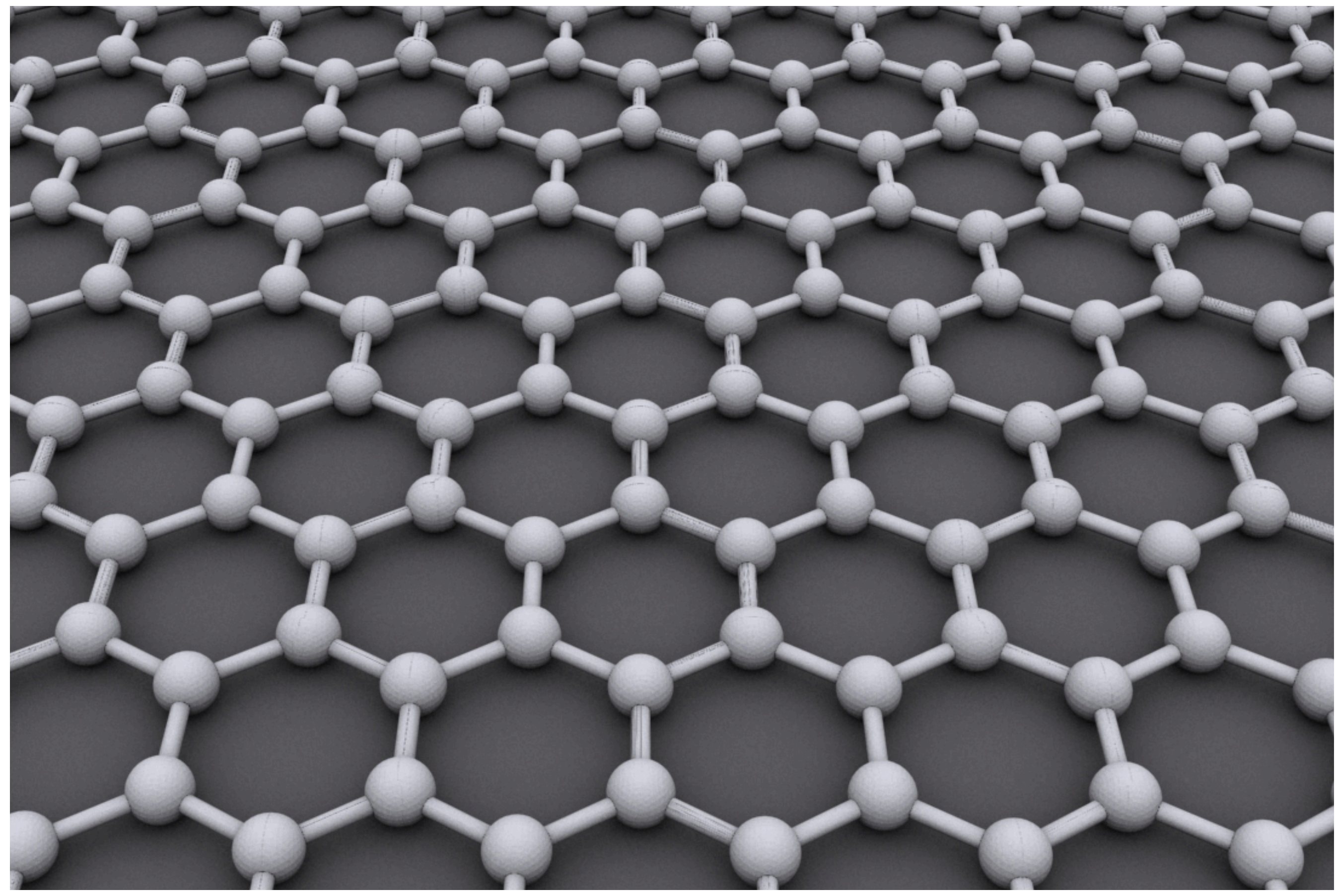
LTM: from least connected to most connected

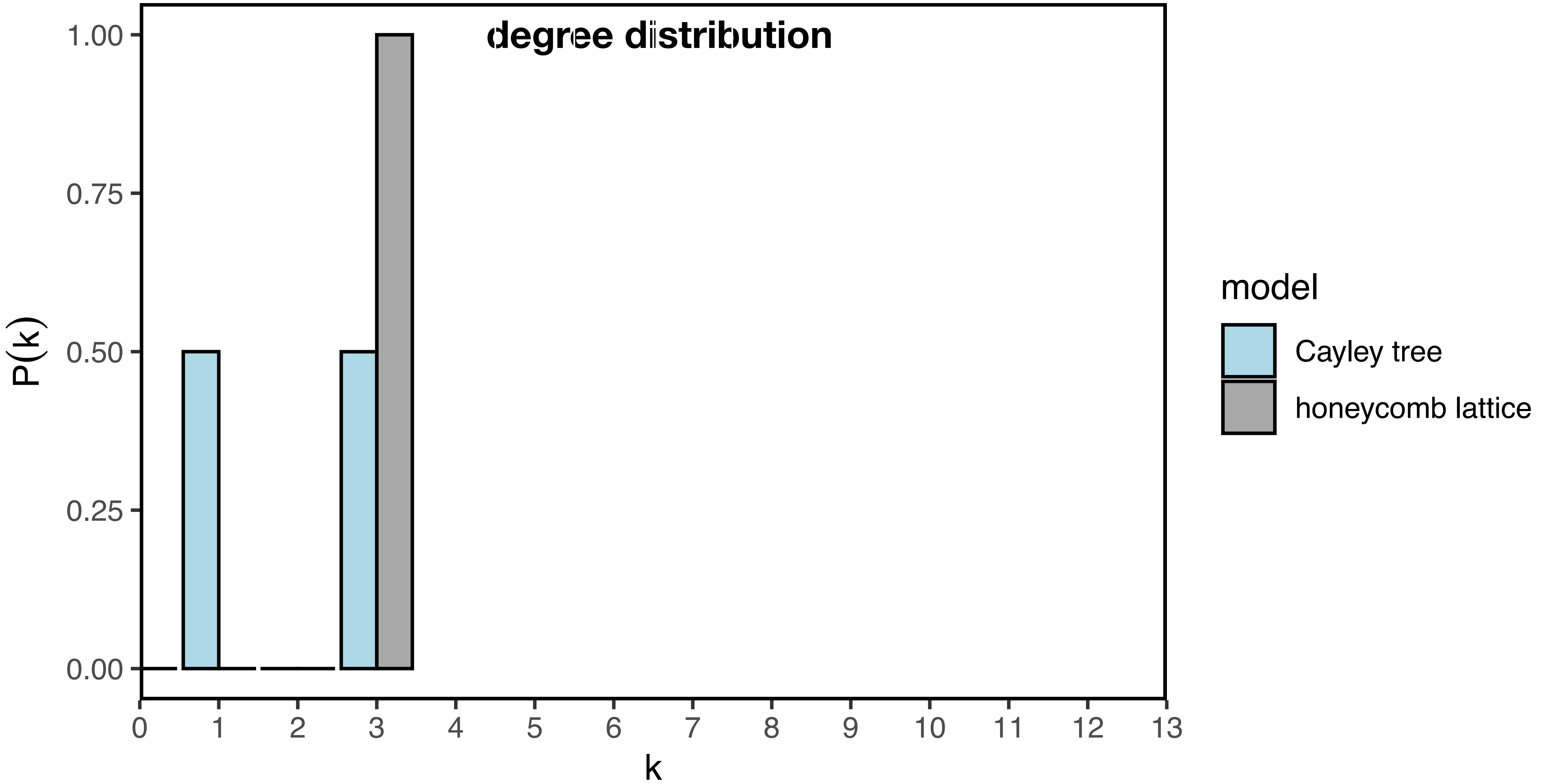


# honeycomb lattice

named after bees…

… but we think of Graphene





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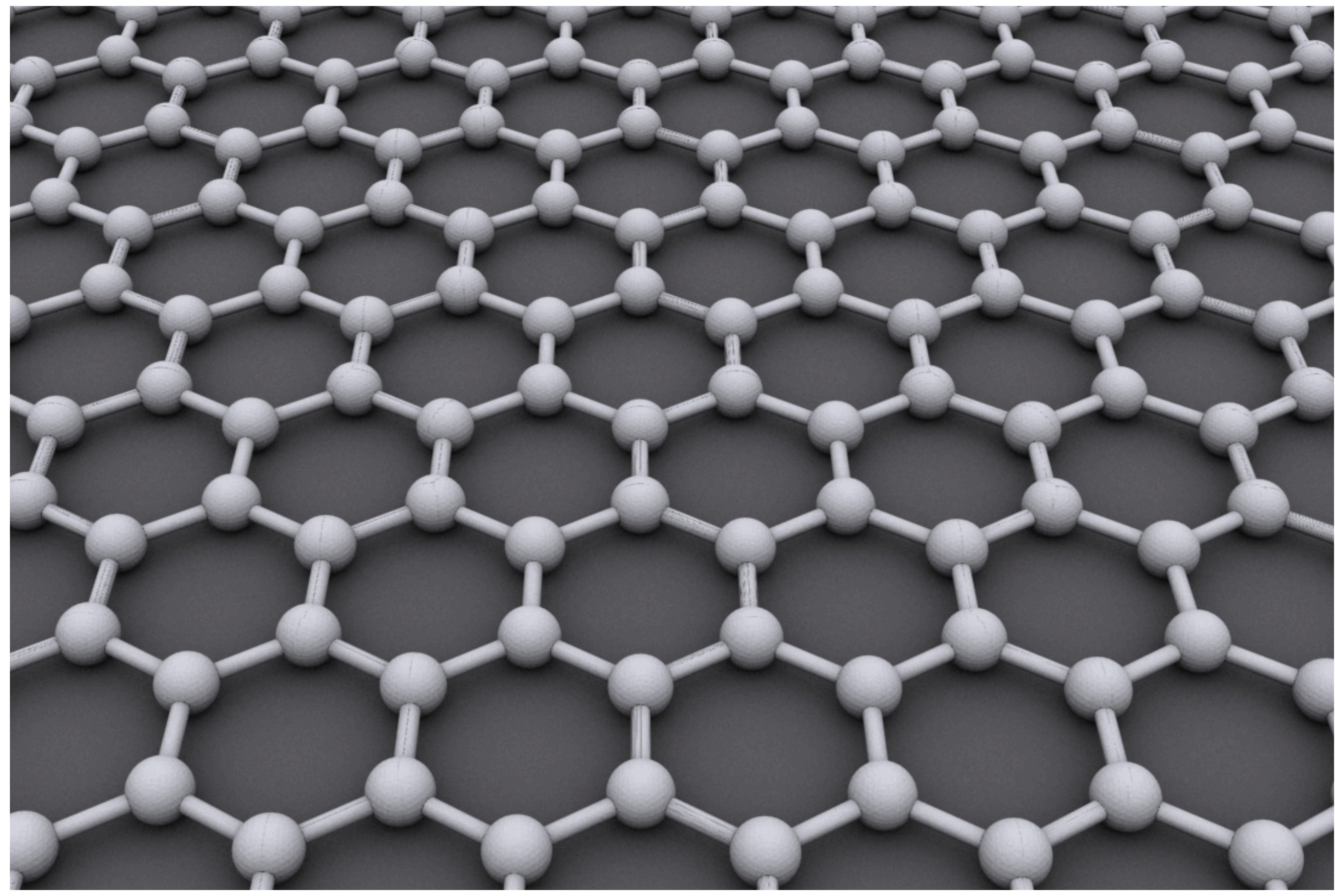
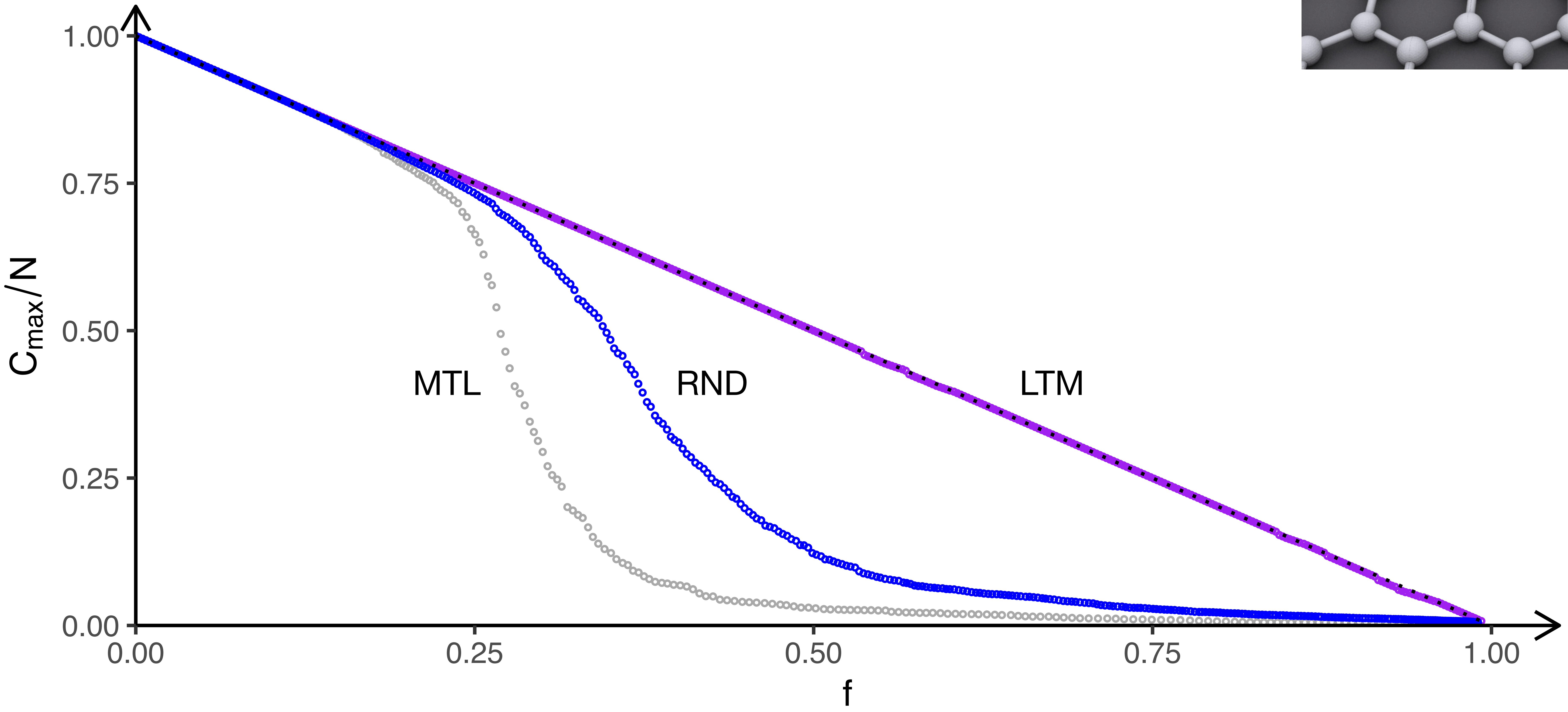
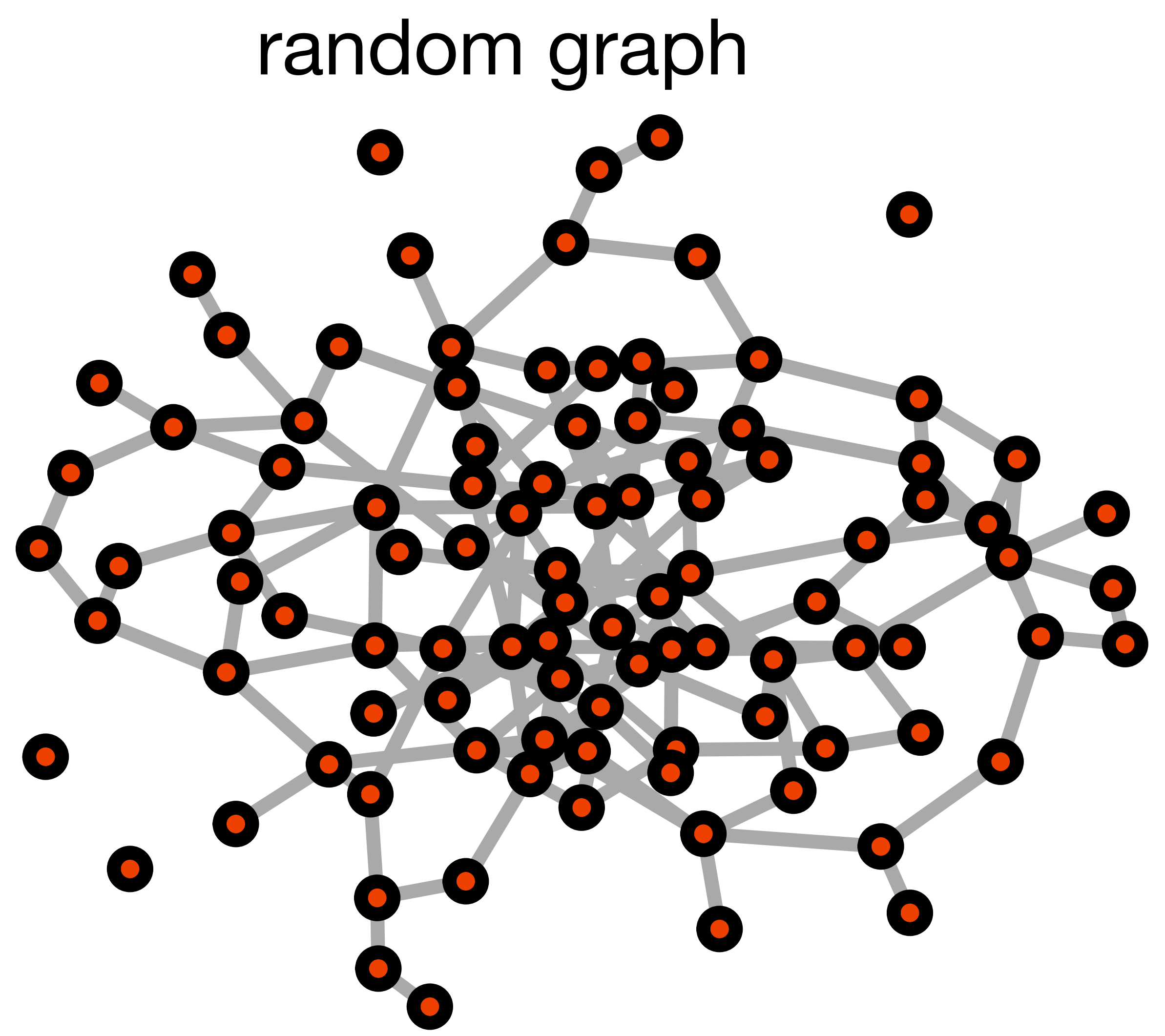


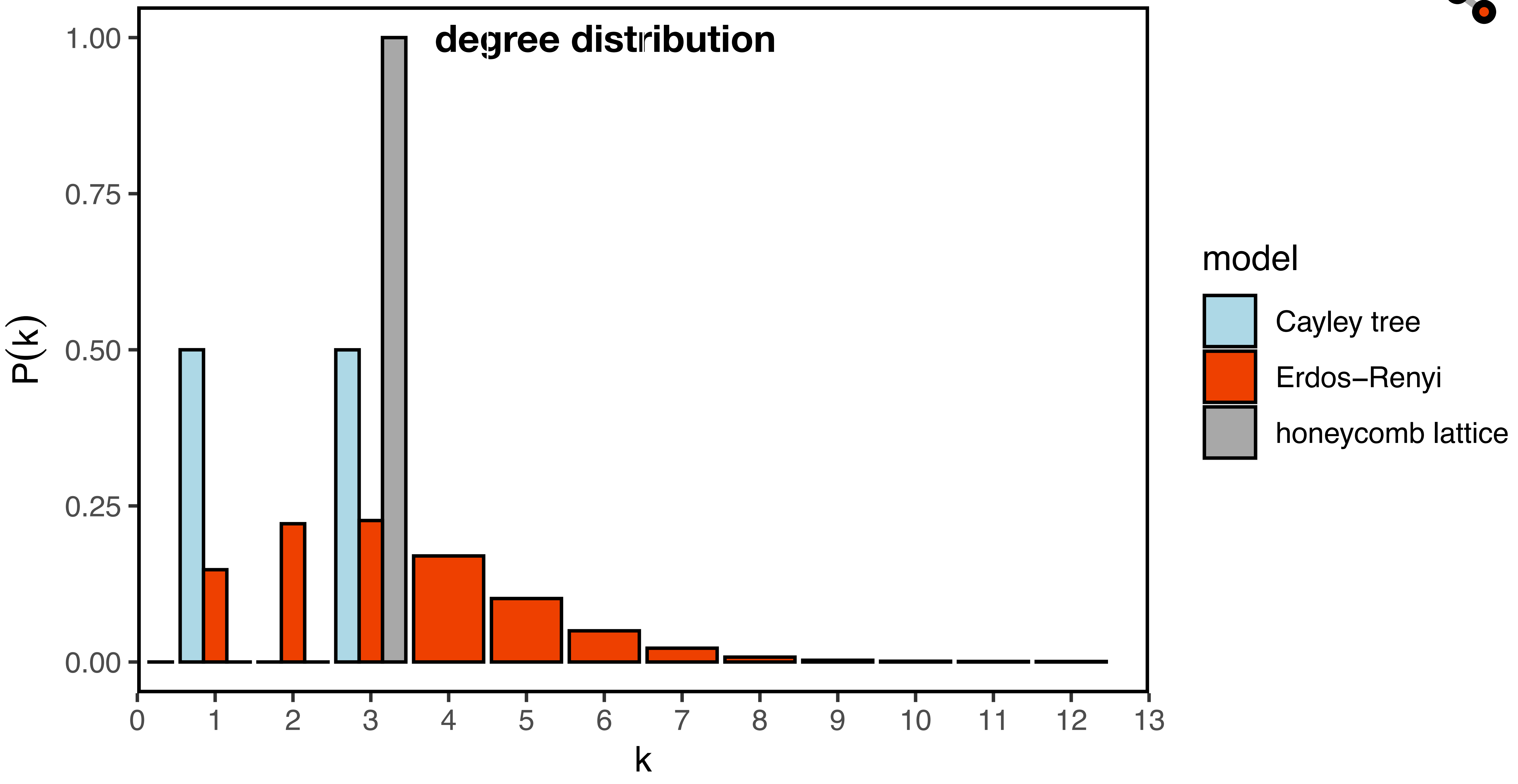


# Erdos-Renyi graph

$$P(k) = \binom{N-1}{k} p^k (1-p)^{N-1-k}$$

binomial degree distribution



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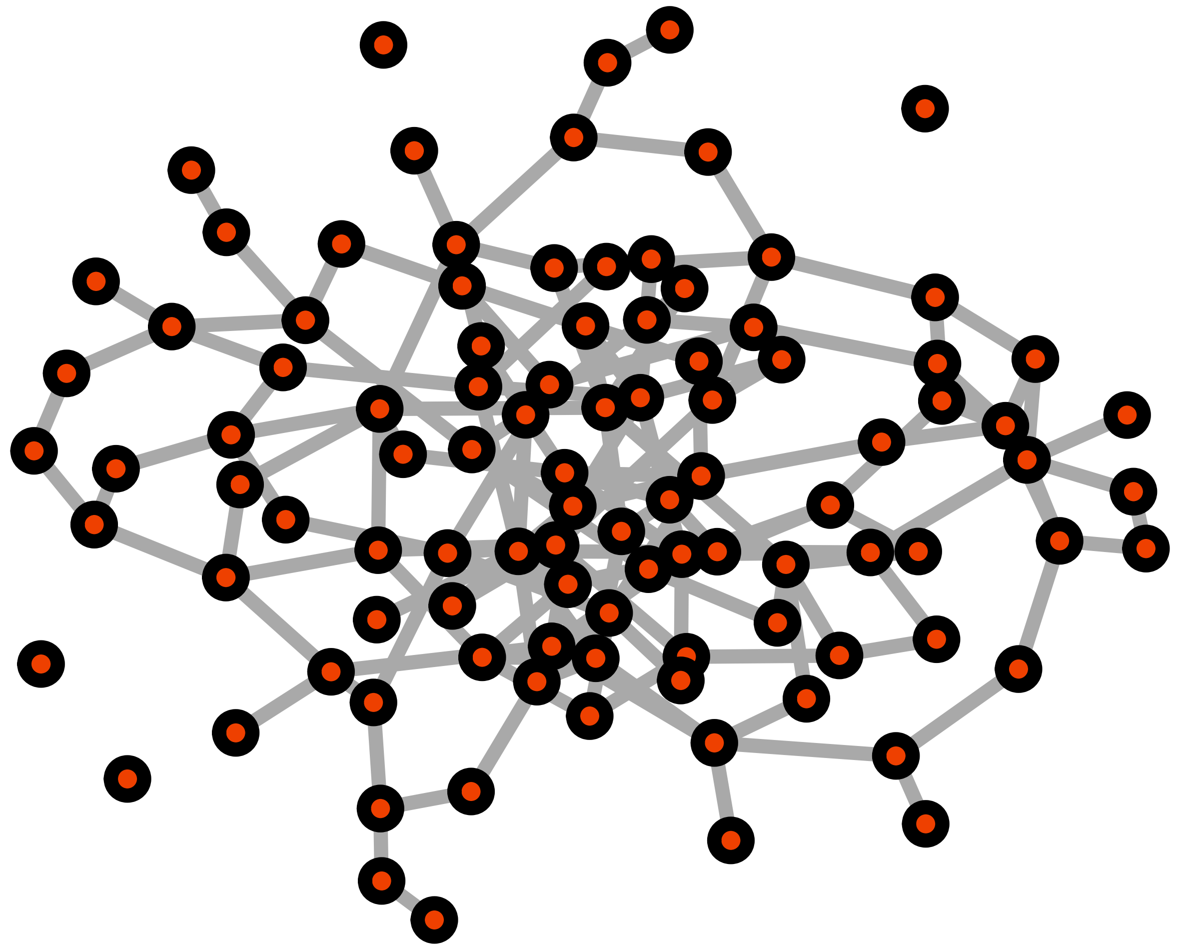
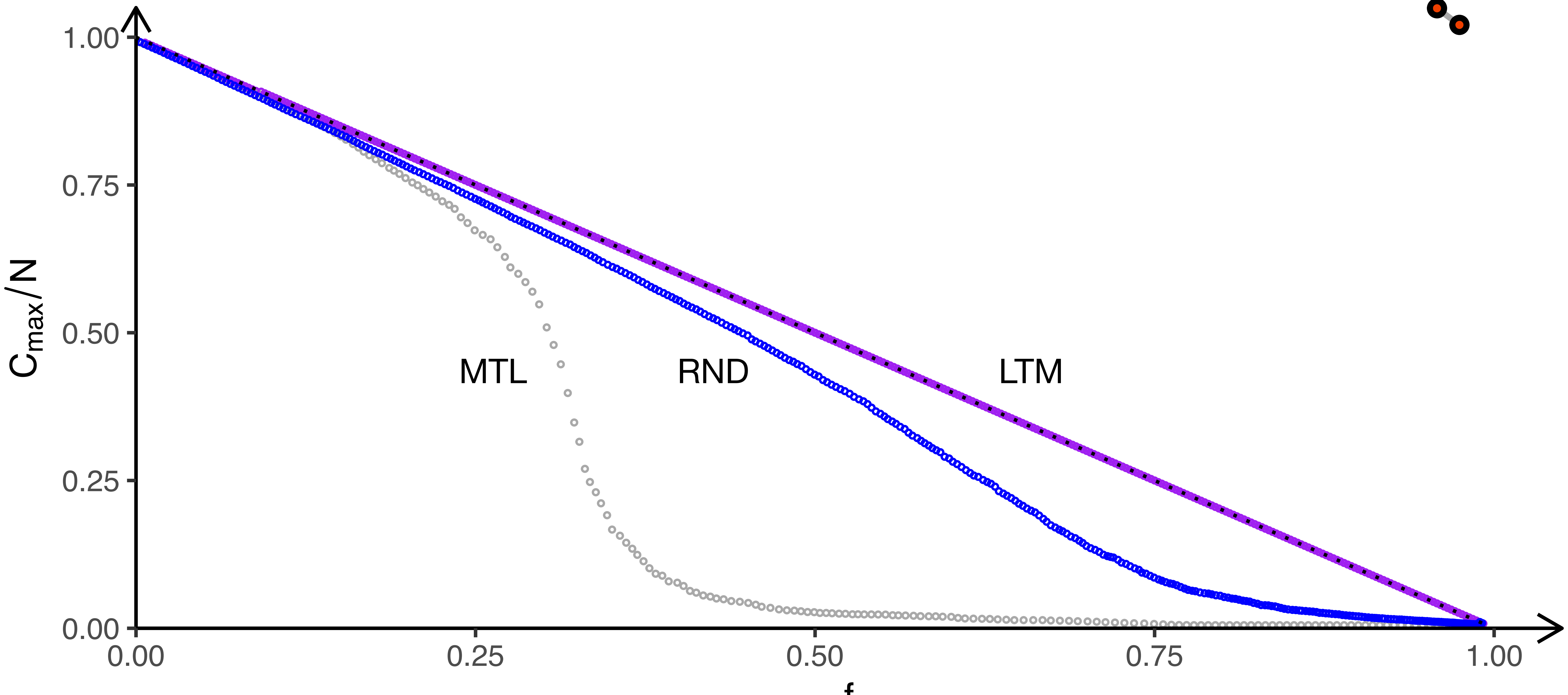
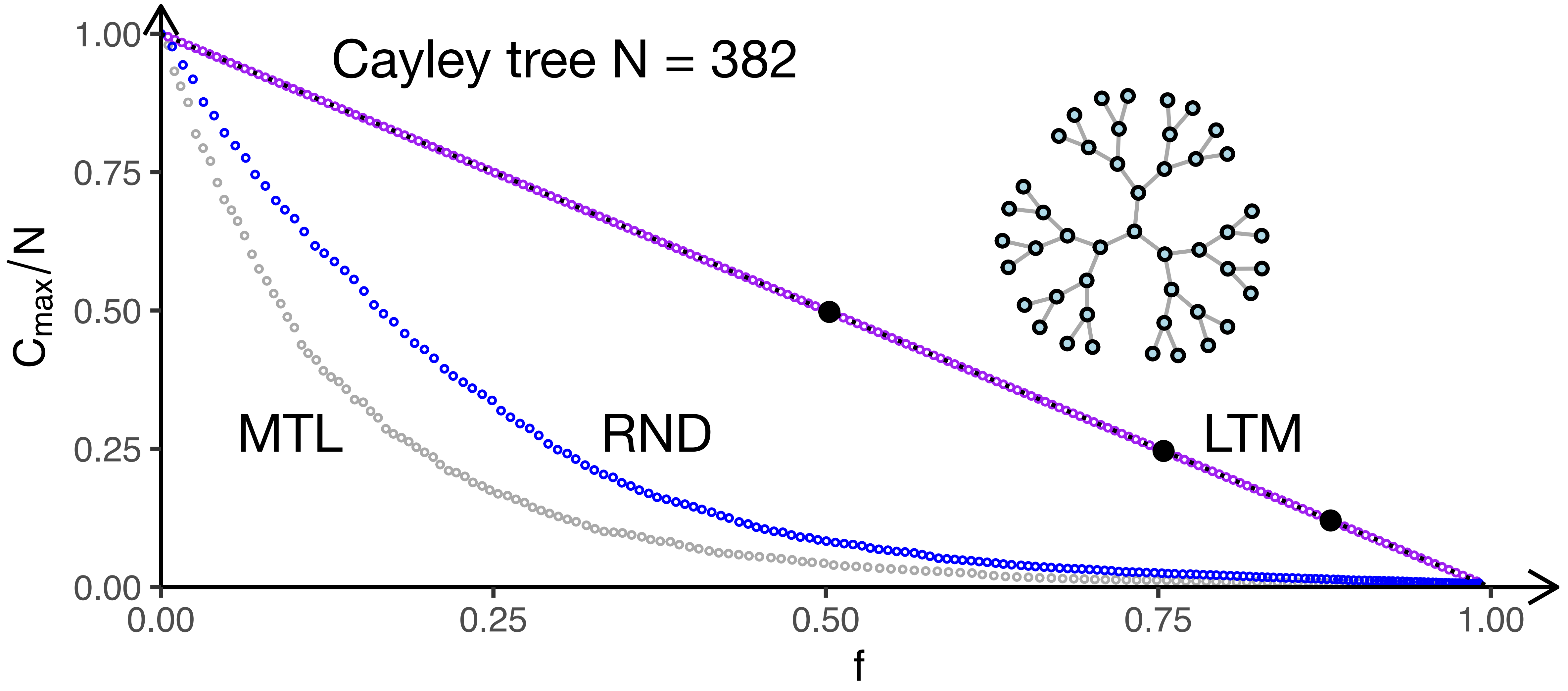
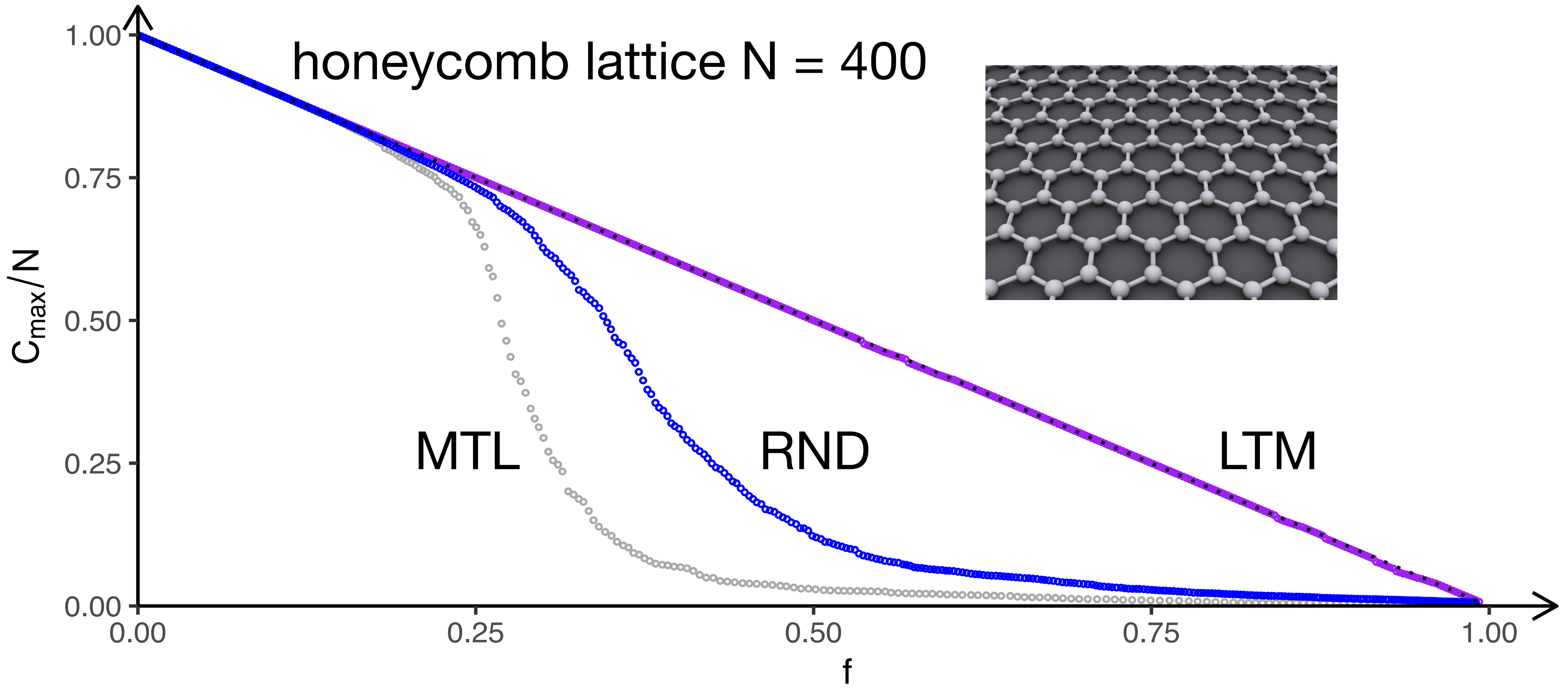
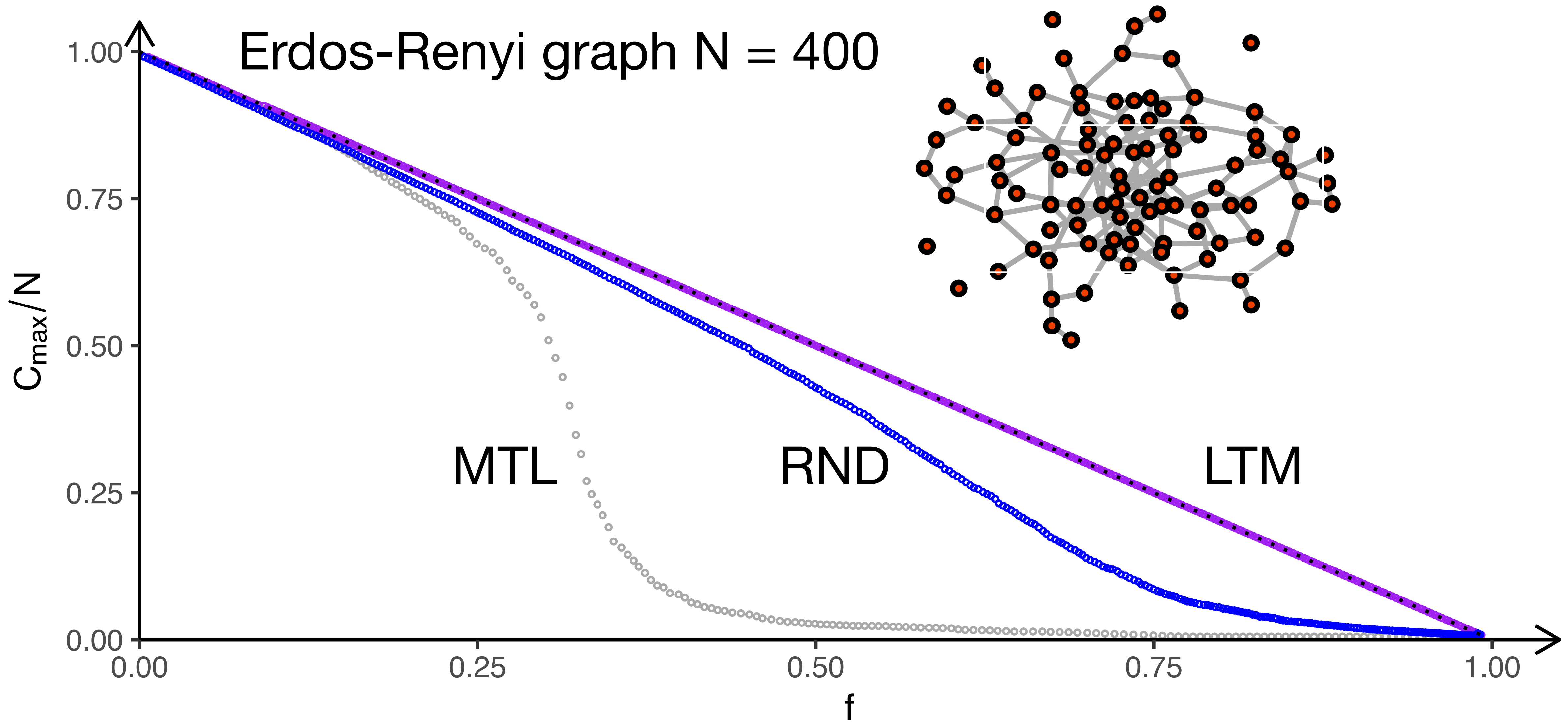


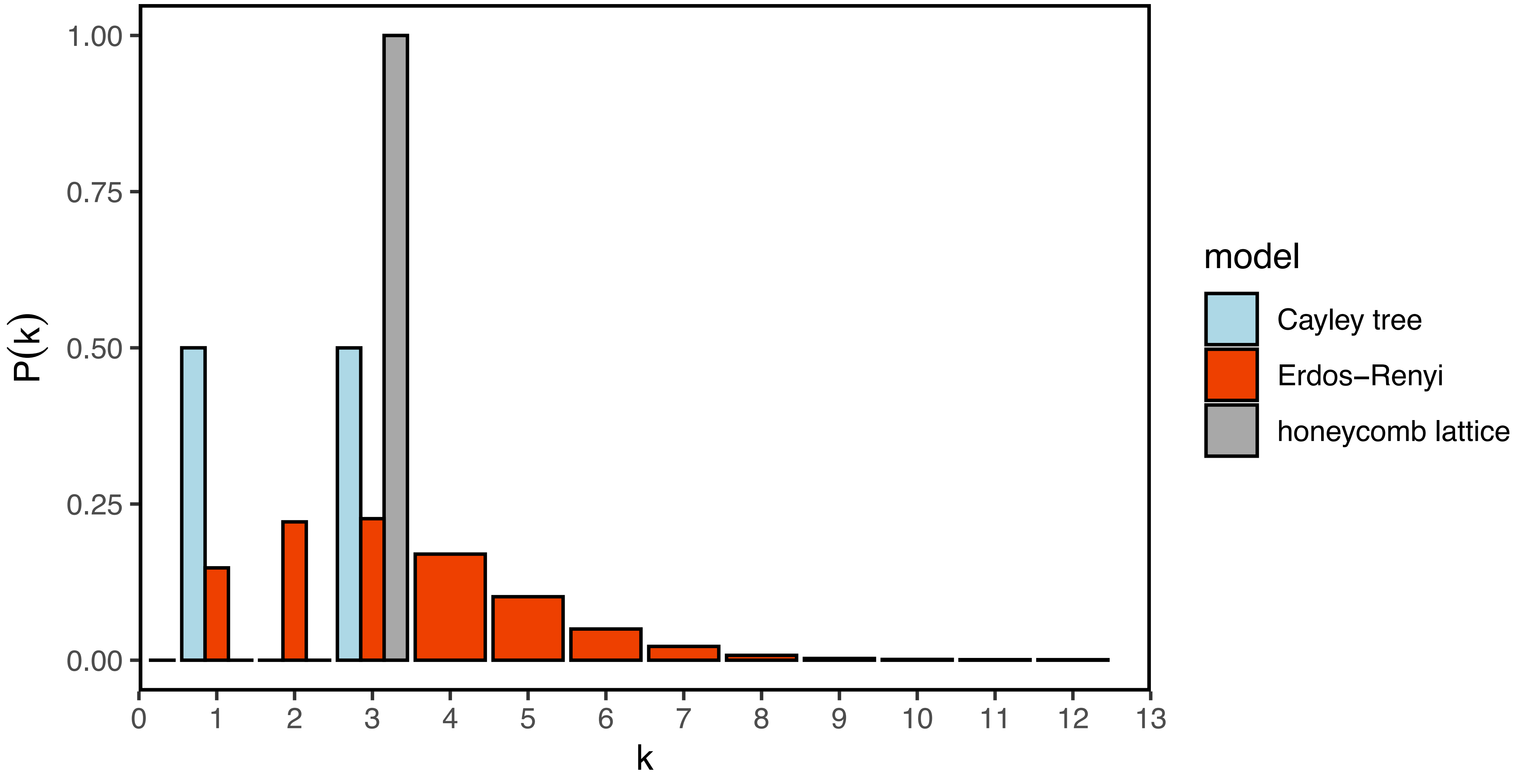


# …discriminate

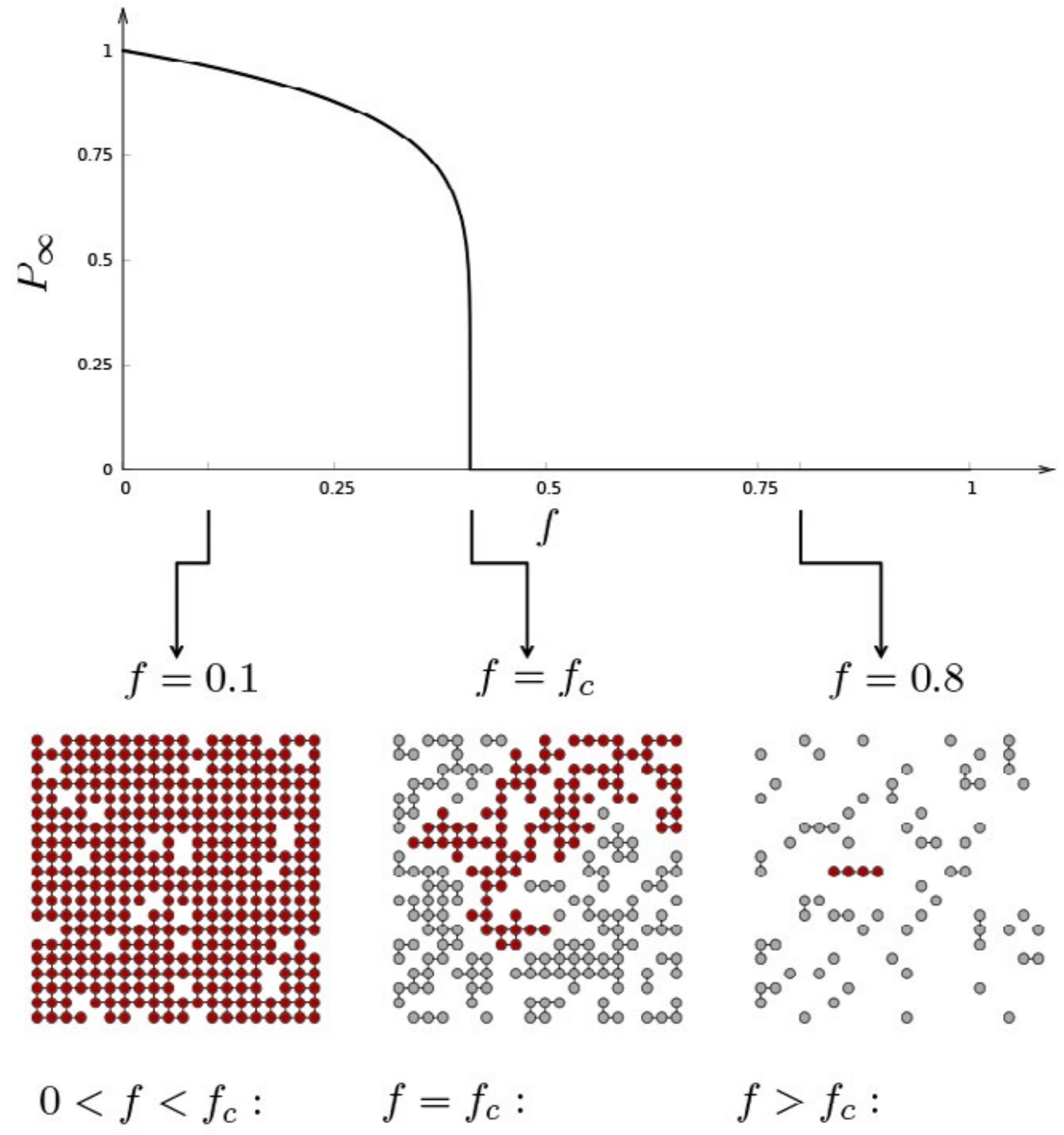


# percolation theory




$f$ = probability of a site to be empty

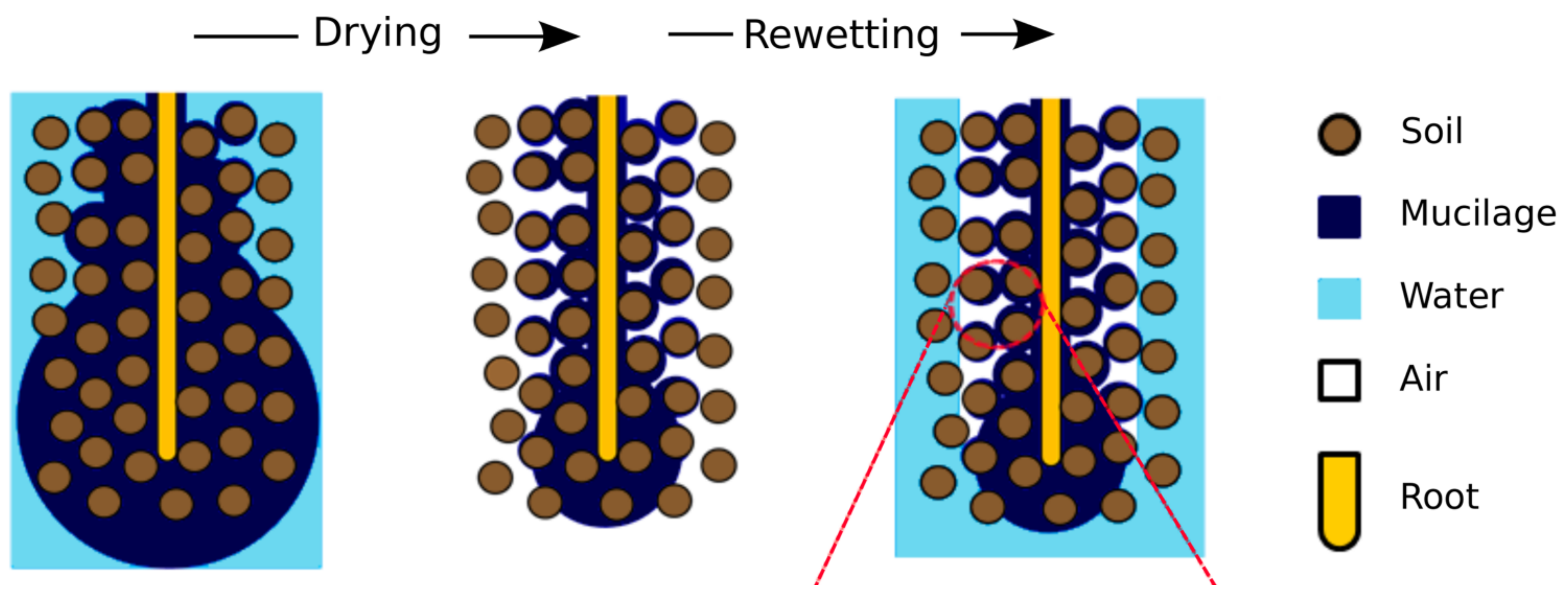
$f$ = 0 => all sites in the lattice are occupied

$f$ = 1 => all sites in the lattice are empty

$P_\infty$ = probability that an occupied a site belongs to the giant cluster

$$P_\infty = \frac{C_{max}}{N}$$ in the following

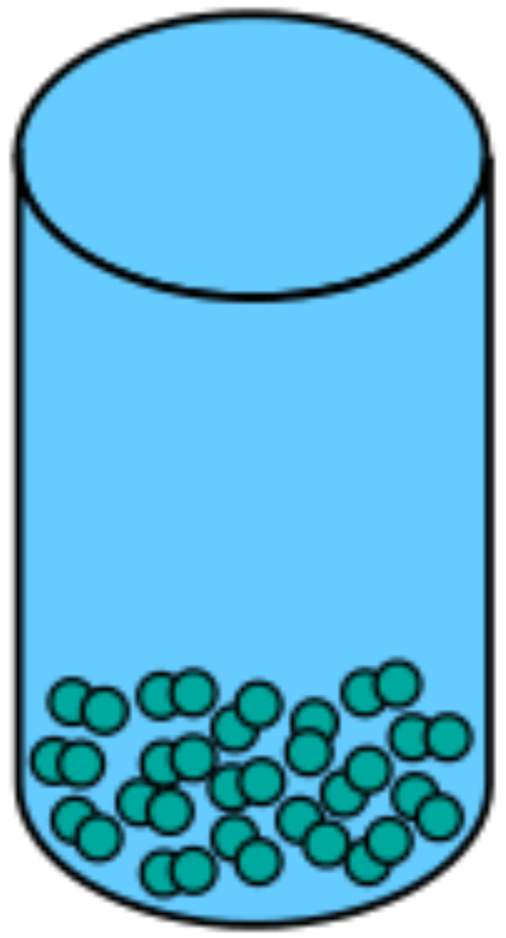
**e.g. percolation applied to hydrology**

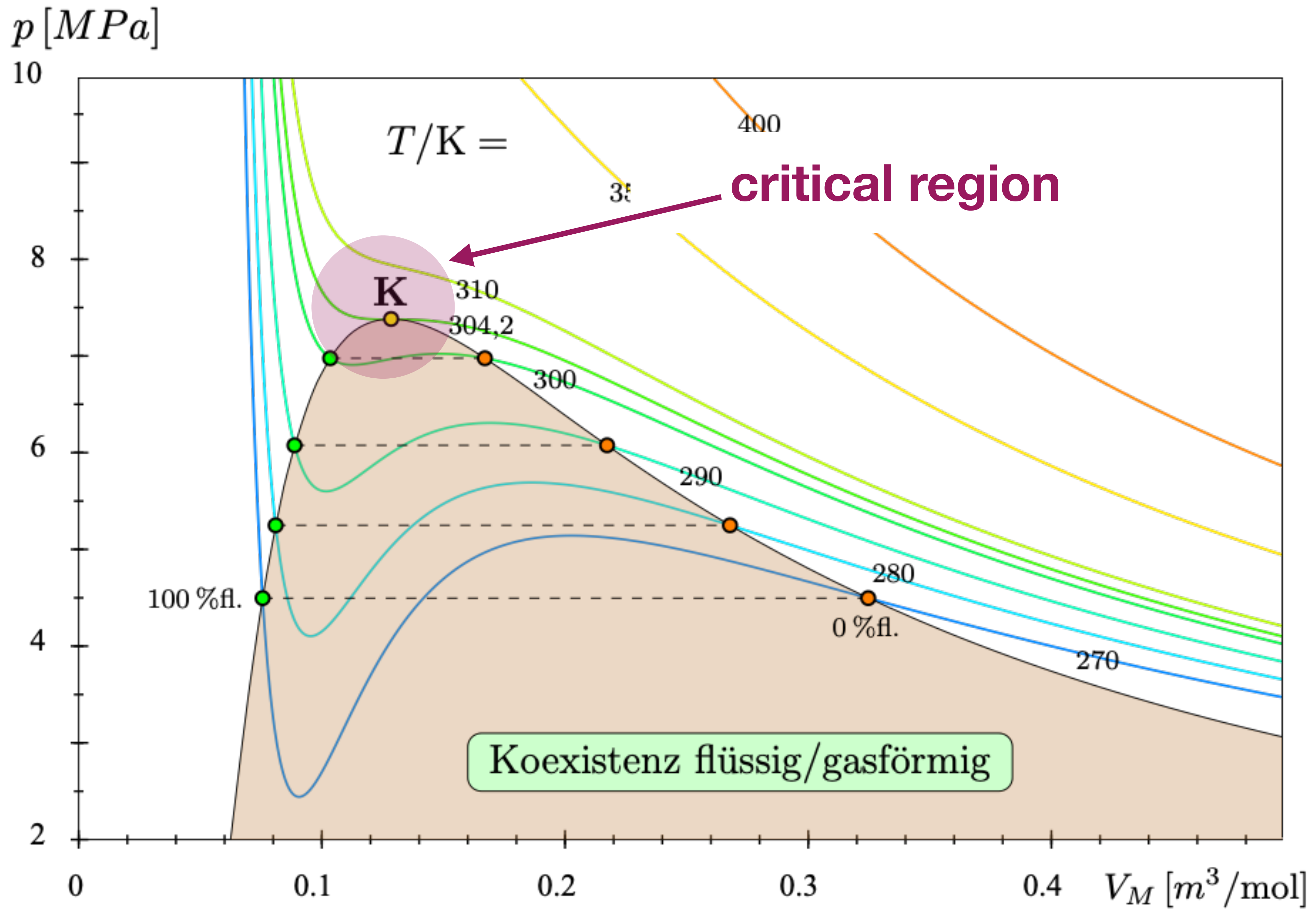



Kroener, E. *et al.* (2015). Phys Rev. E **91**, 042706.

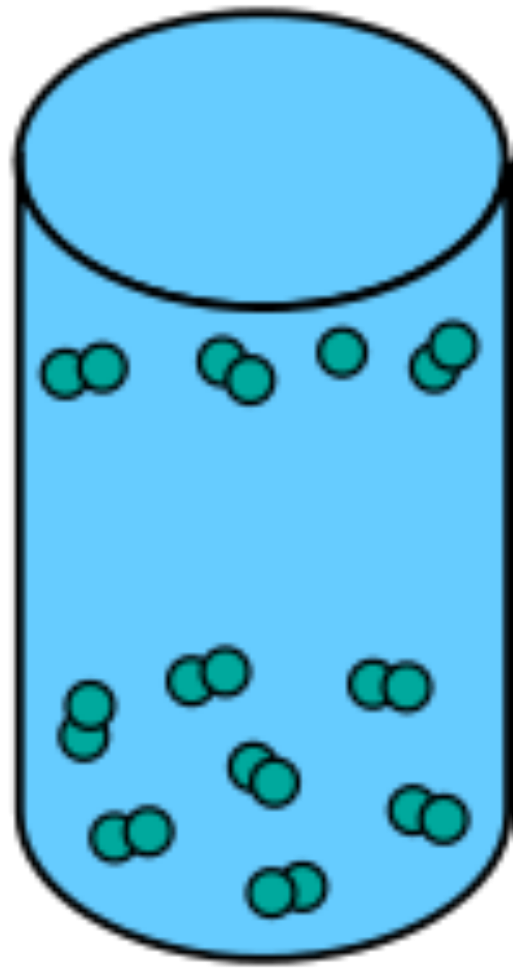
# percolation is actually a critical phase transition

**liquid**

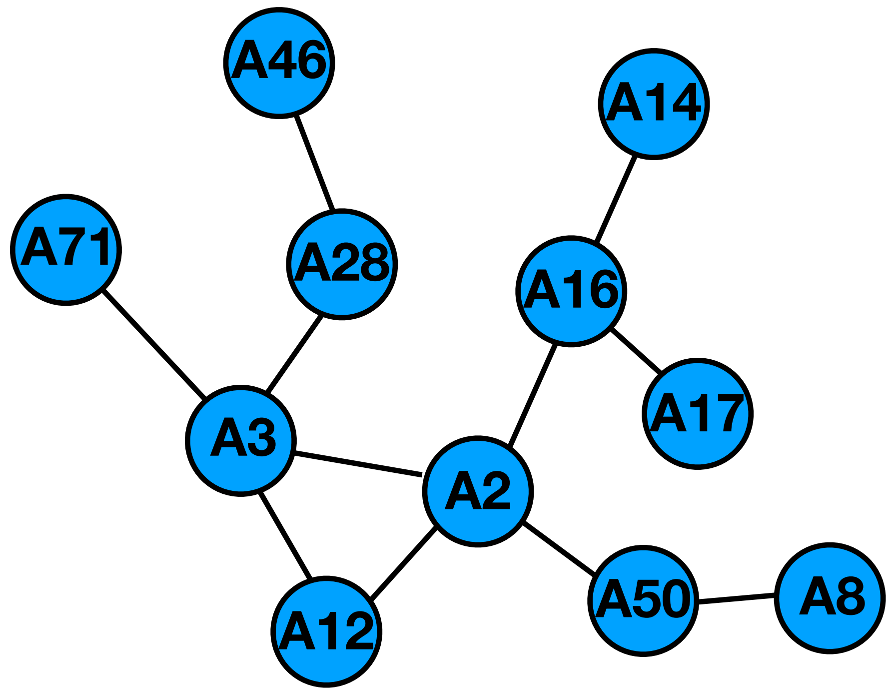





**gas**



**connected phase**

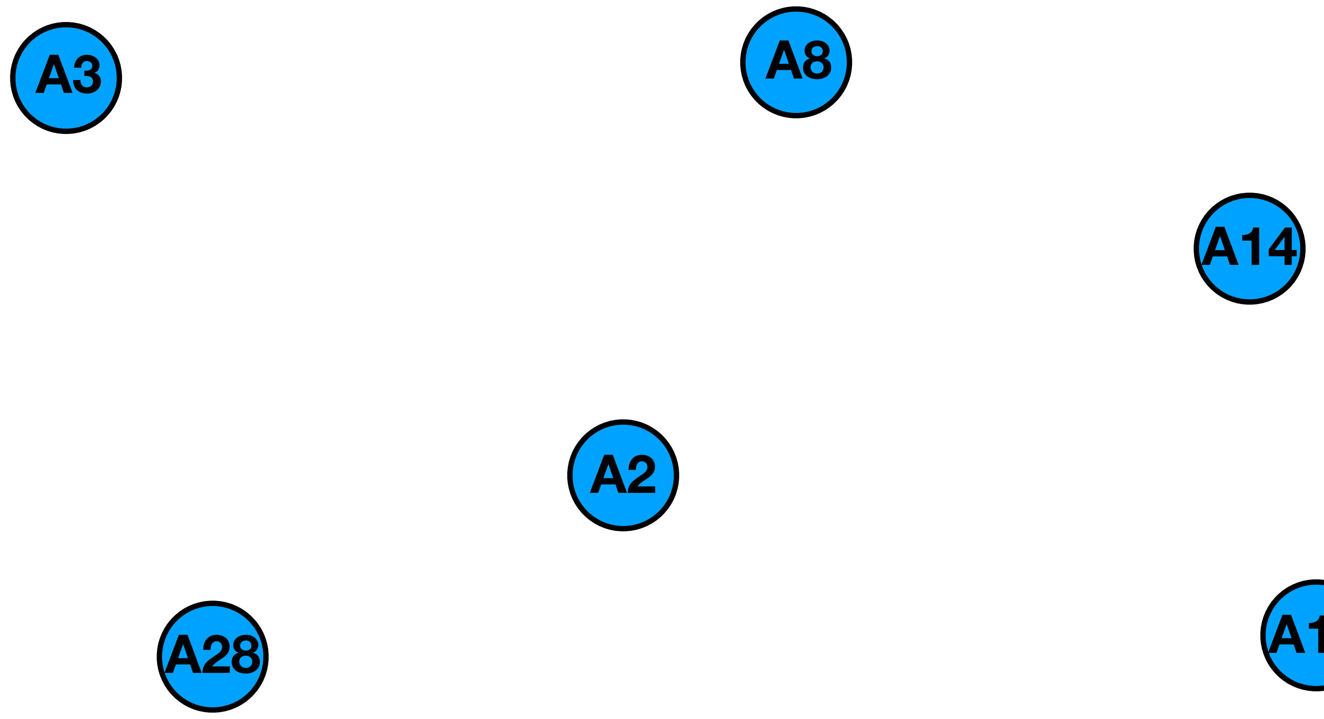



“phase of matter”

“crystal structure”

**degree distribution** $P(k)$

**disconnected phase**



## critical phase transitions in physics

liquid — **critical region** — gas

ferromagnet — $T_c$ — paramagnet — $T$

$$\frac{\rho_l - \rho_g}{\rho_c} \sim \left(\frac{T_c - T}{T_c}\right)^{\beta}$$

---

## percolation phase transition in networks

**supercritical/connected phase** — **critical region** — **subcritical/disconnected phase**

$C_{max} \sim N^{\Delta_c}$ — $f_c$ — $\frac{C_{max}}{N} \to 0$ for $N \to \infty$ — $f$

$N$ = number of nodes in the network

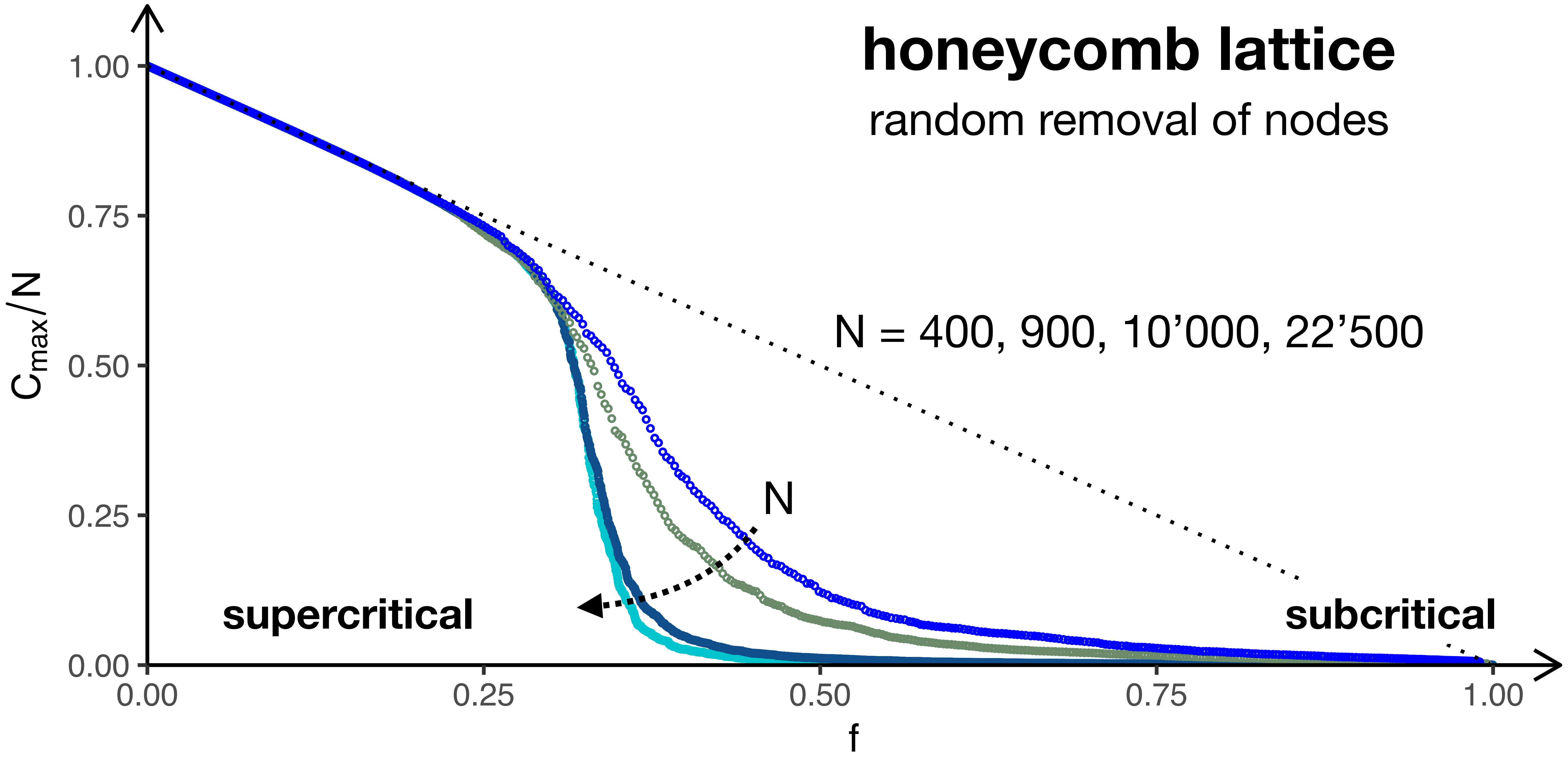
$P_\infty = \frac{C_{max}}{N}$ is a good **order parameter**

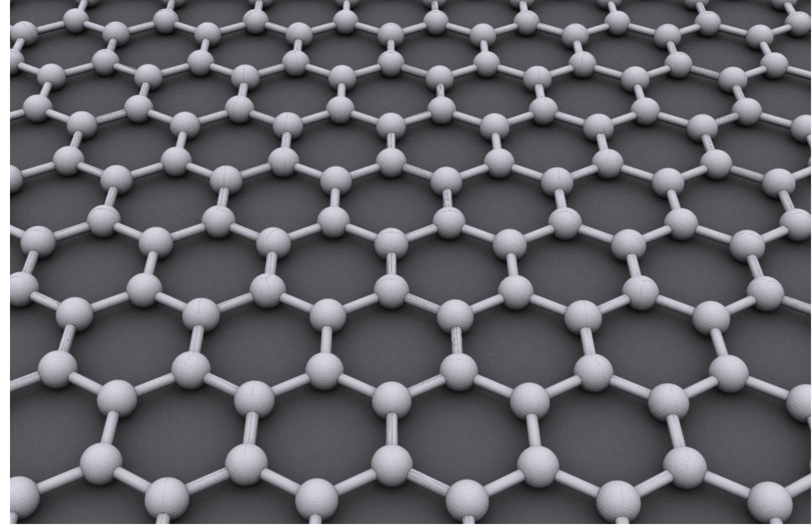
$y = \frac{C_{max}}{N^{\Delta_c}}$ vs $x = \frac{f - f_c}{N^{\Delta_f}}$ **master curve**

$\Delta_c$ and $\Delta_f$ depend on the **topology** of the network

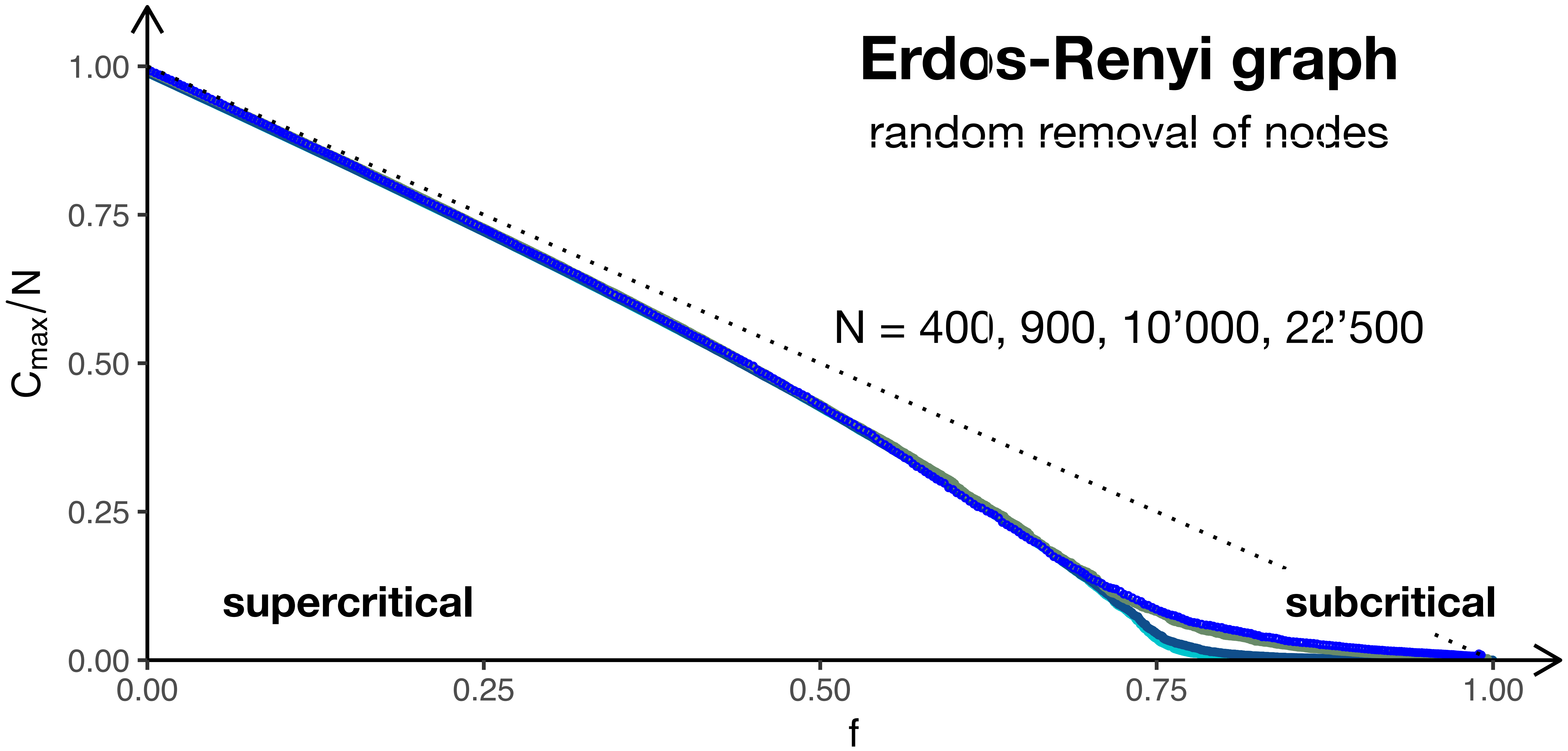
| **model** | $\Delta_c$ | $\Delta_f$ |
|---|---|---|
| Erdos-Renyi | 2/3 | 1/3 |
| 2D percolation | 303/288 | 3/8 |

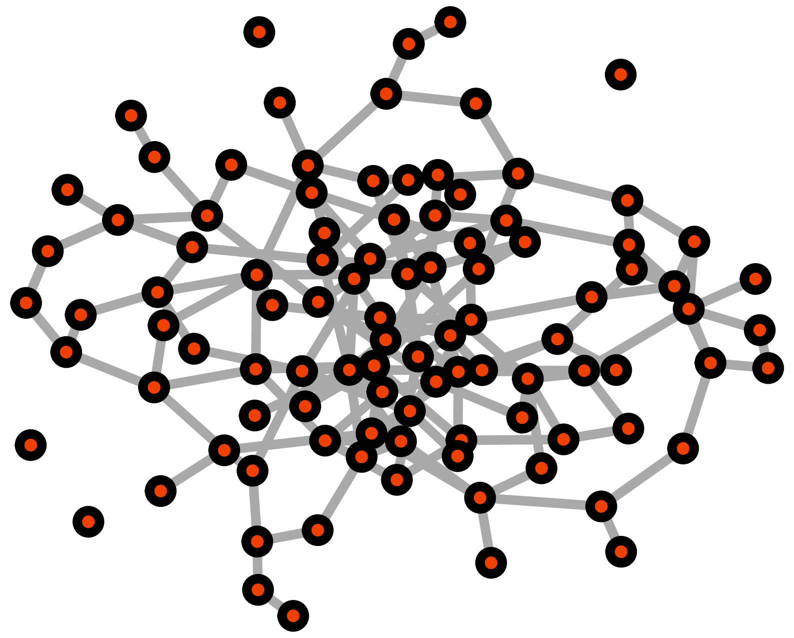
see e.g. Guimaraes, A.-L. (2020). Ann. Rev. Ecol. Evol. Syst. 51: 433-60



increasing the number of nodes N the transition gets progressively sharper and approaches the **2D percolation** result



increasing the number of nodes N the transition gets progressively sharper as expected for the **Erdos-Renyi graph**

# Molloy-Reed criterion

# Molloy-Reed criterion

valid for every random network

**supercritical/connected phase** — **subcritical/disconnected phase**

$C_{max} \sim N^{\Delta_c}$ $\qquad f_c \qquad$ $\frac{C_{max}}{N} \to 0$ for $N \to \infty$ $\qquad f$

$\kappa \geq 2$ $\qquad\qquad$ $\kappa < 2$

$$\kappa = \sum_i k_i P(k_i | i \leftrightarrow j)$$

$P(A|B)$ **conditional probability**

A: i-th node has degree k

B: i-th node is connected to the j-th node

generally $\kappa(f)$ therefore we can define the critical threshold as $\kappa(f_c) = 2$

in terms of the original distribution $f_c = 1 - \frac{1}{\kappa - 1}$

Molloy, M. and Reed, B. (1995).Random Struct. Algorithms **6**, 161.

# Molloy-Reed criterion

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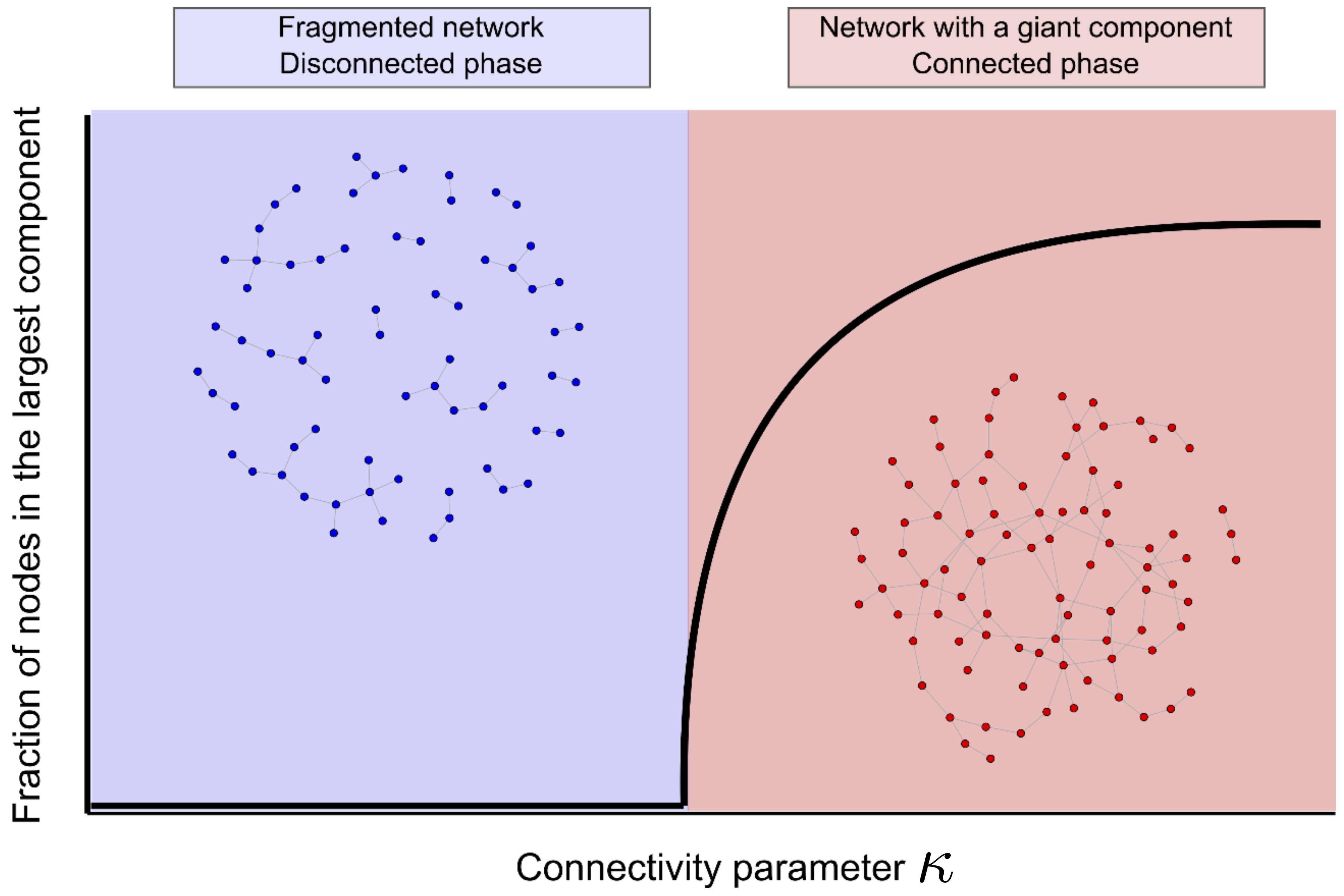


Molloy, M. and Reed, B. (1995).Random Struct. Algorithms **6**, 161.

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**Erdos-Renyi graph**

$$P(k) = \binom{N-1}{k} p^k (1-p)^{N-1-k}$$

binomial degree distribution

$$\langle k^2 \rangle - \langle k \rangle^2 = \langle k \rangle \quad \Rightarrow \quad \langle k^2 \rangle = \langle k \rangle^2 + \langle k \rangle$$

property binomial distribution

$$\kappa = \frac{\langle k^2 \rangle}{\langle k \rangle} = \langle k \rangle + 1 \geq 2 \quad \Rightarrow \quad \boxed{\langle k \rangle \geq 1}$$

known result

---

**scale-free network**

$$P(k) = Ck^{-\gamma}$$

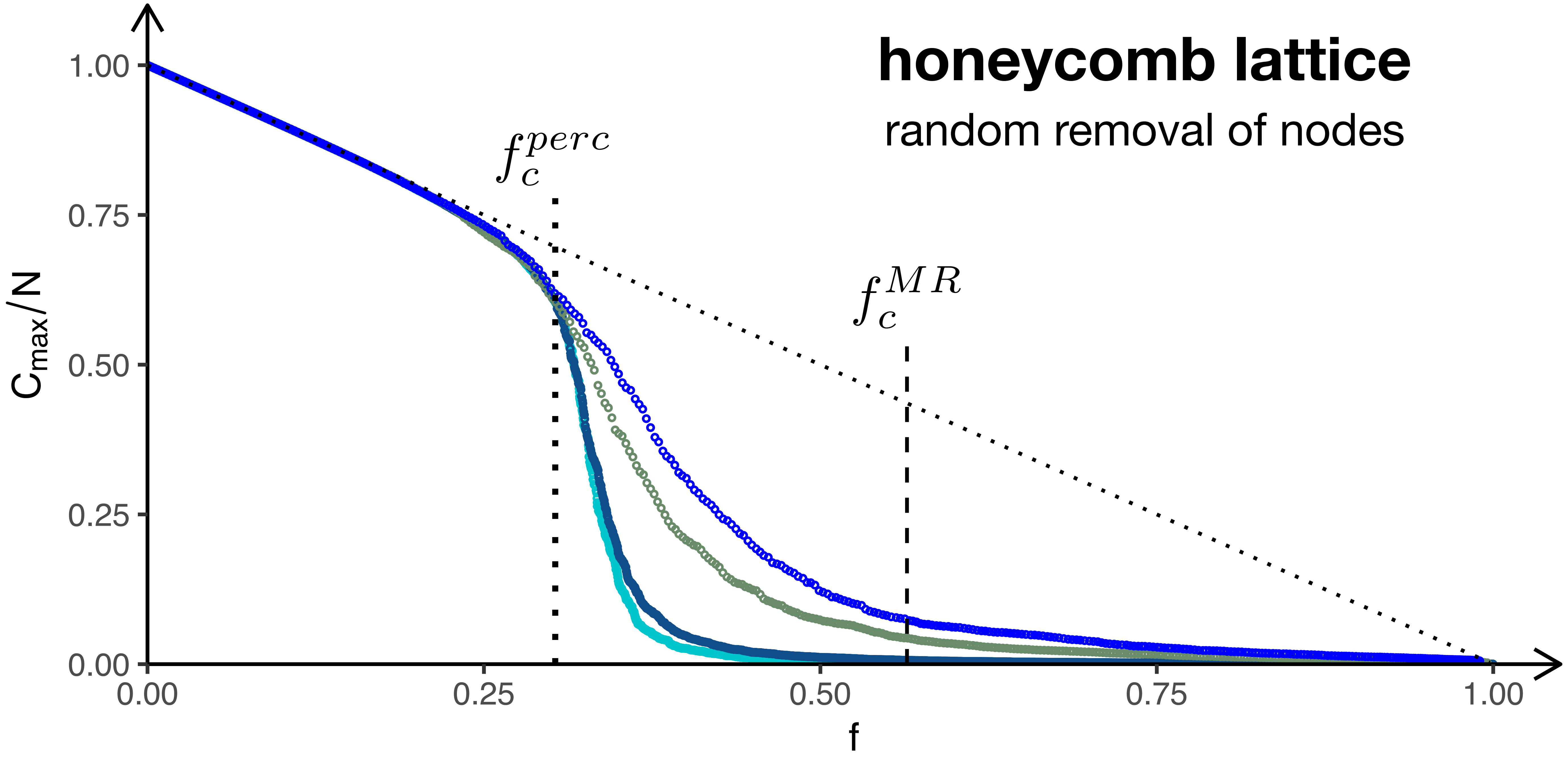
$C$ generally depends on $k_{min}$ and $k_{max} \sim N^{1/(\gamma-1)}$

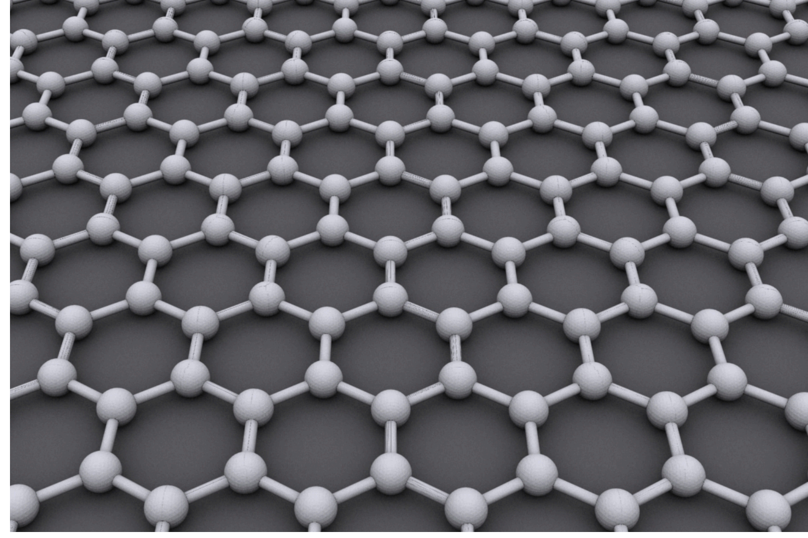
in the limit $N \to \infty$

| | |
|---|---|
| $f_c \to 1$ | for $2 < \gamma \leq 3$ |
| $f_c < 1$ | for $\gamma > 3$ |

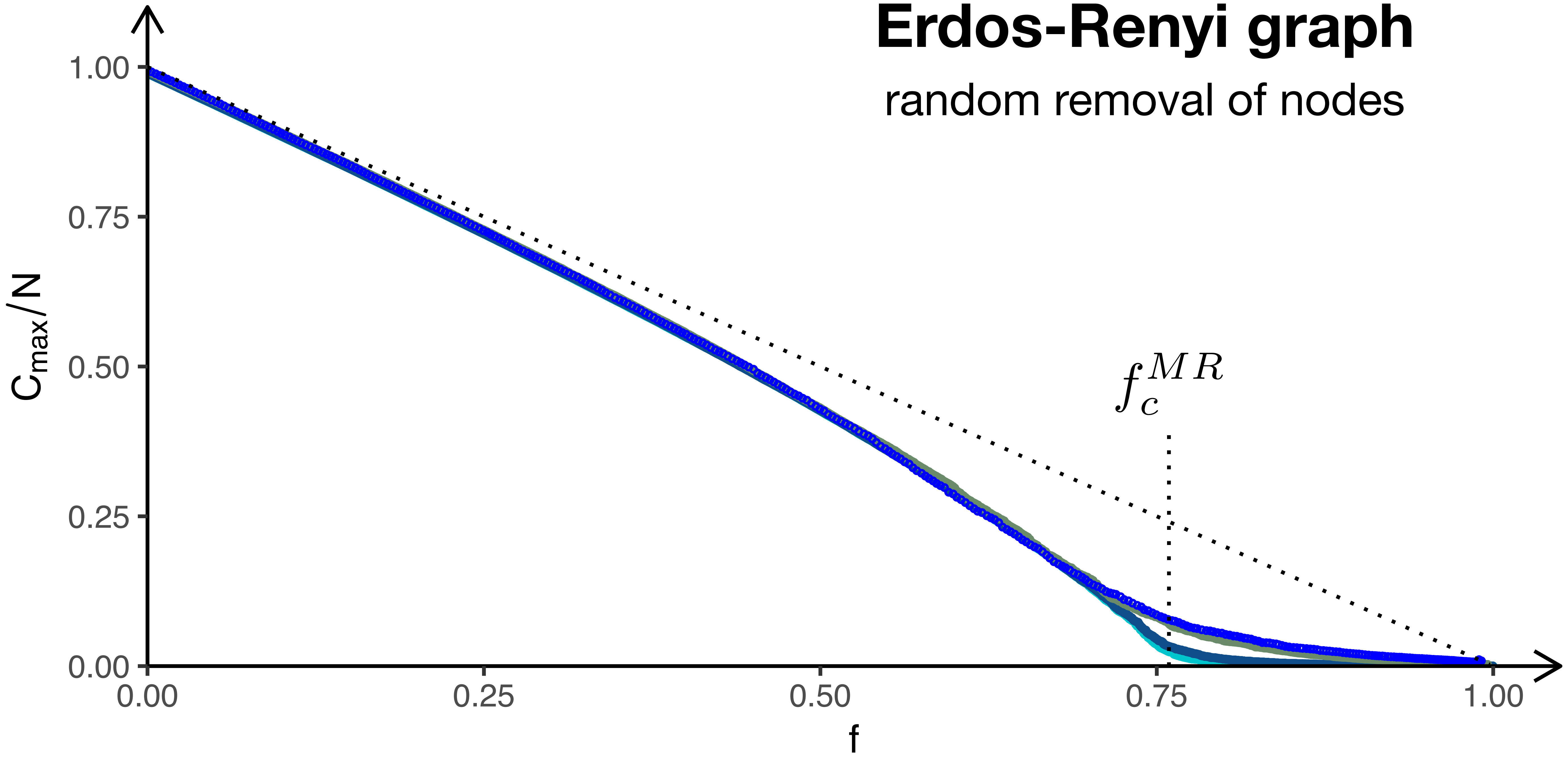
# Molloy-Reed criterion

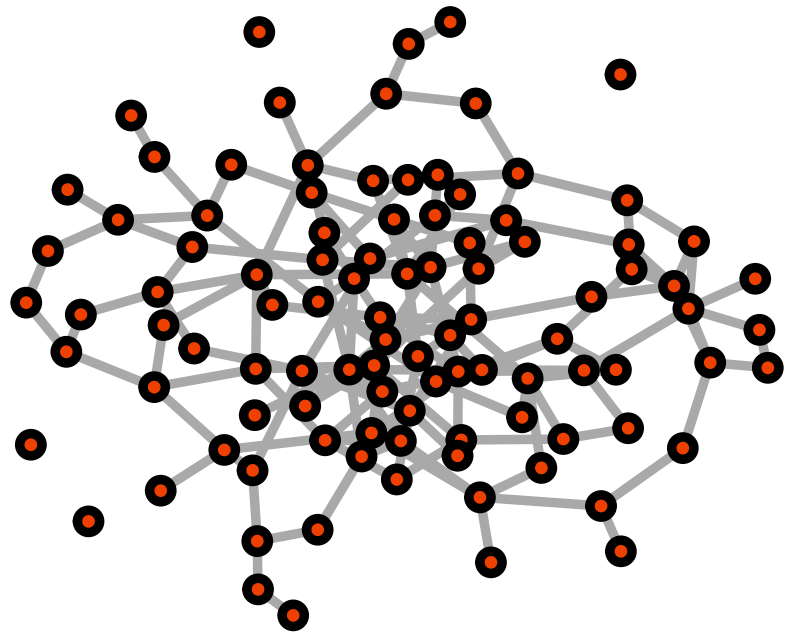
for random scale-free networks $P(k) = Ck^{-\gamma}$



the **Molloy-Reed** criterion **fails** to predict the correct critical fraction; **2D percolation** is instead accurate



the **Molloy-Reed** criterion predicts the correct critical fraction

Molloy-Reed critical threshold in different models
N = 400, 900
honeycomb lattice
Erdos-Renyi graph
scale-free NW, gamma = 2.5
scale-free NW, gamma = 3.5
MTL
RND
$C_{max}/N$
f
0.00
0.25
0.50
0.75
1.00

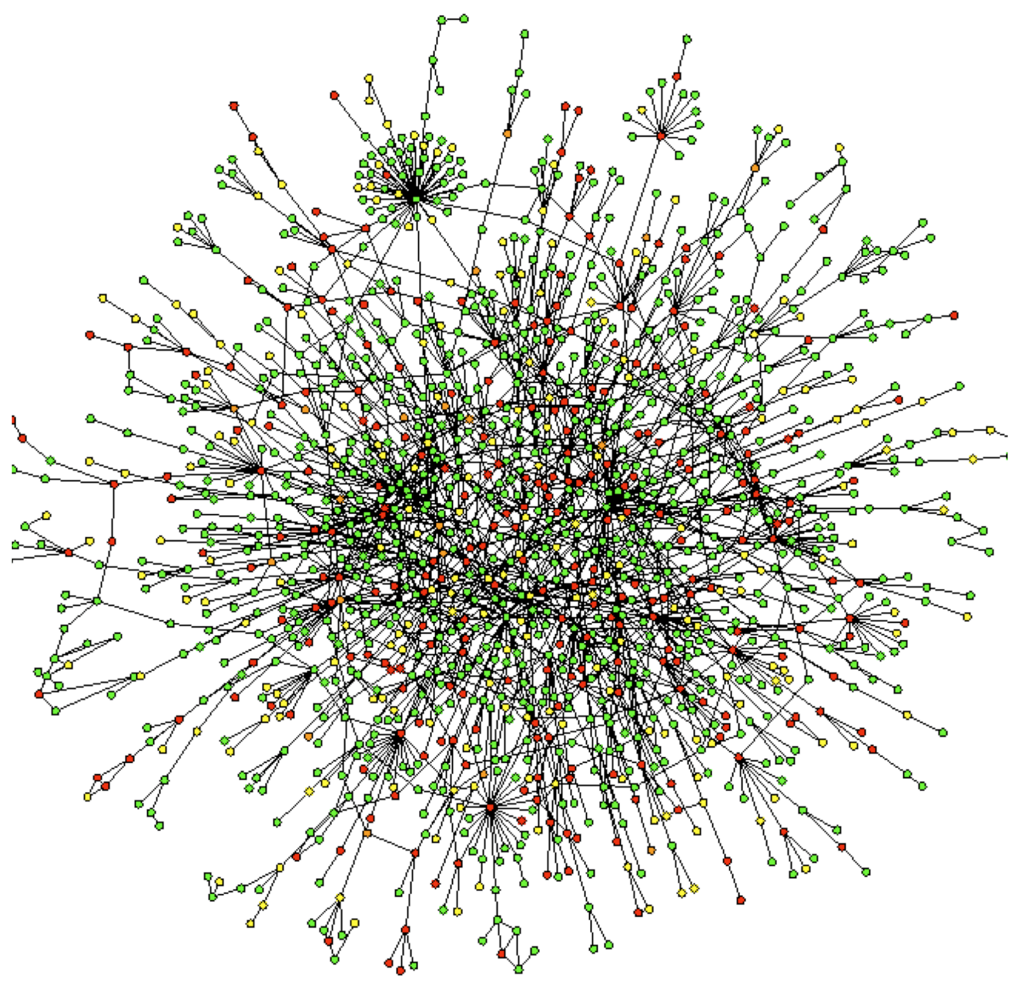
## robustness of real networks to random removal of nodes

$$f_c = 1 - \frac{1}{\kappa - 1}$$

| **network** | $N$ | $L$ | $\langle k \rangle$ | $\langle k^2 \rangle$ | $\gamma$ | $\kappa$ |
|---|---|---|---|---|---|---|
| www-in | 325'729 | 1'497'134 | 4.6 | 1'546 | 2 | 336 |
| www-out | 325'729 | 1'497'134 | 4.6 | 482 | 2.31 | 105 |
| email-in | 57'194 | 103'521 | 1.81 | 1'546 | 3.43 | 19 |
| email-out | 57'194 | 103'521 | 1.81 | 482 | 2.03 | 643 |
| citation-in | 449'673 | 4'689'479 | 10.43 | 971.5 | 3 | 93 |
| citation-out | 449'673 | 4'689'479 | 10.43 | 198.8 | 4 | 19 |
| actor | 702'388 | 29'397'908 | 83.71 | 47'353 | 2.12 | 565 |
| sc. collaboration | 23'133 | 93'439 | 8.08 | 178.2 | 3.35 | 22 |
| Internet | 192'244 | 609'066 | 6.34 | 240.1 | 3.42 | 38 |
| power grid | 4'941 | 6'594 | 2.67 | 10.3 | [Exp.] | 3.86 |

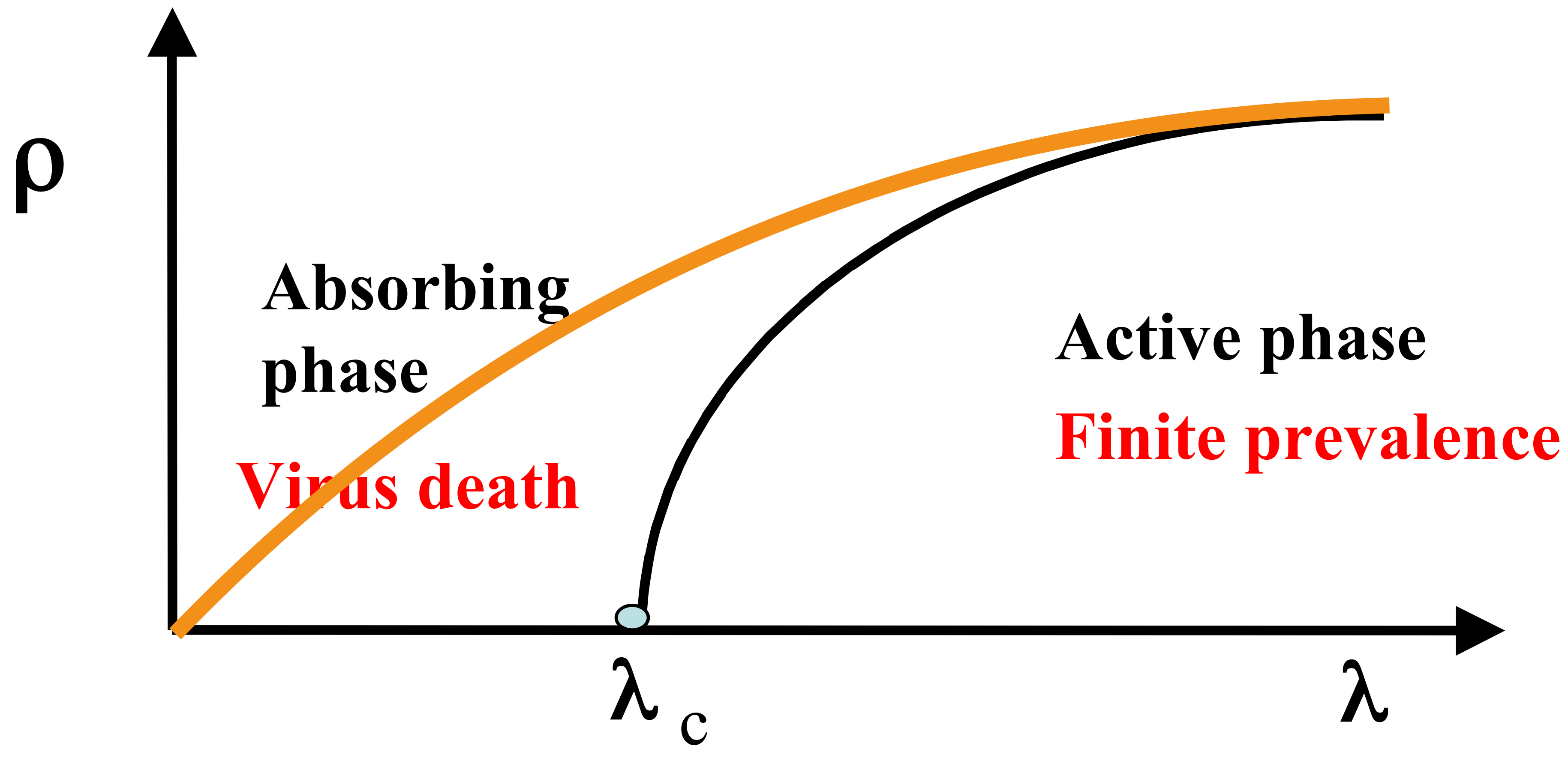

Taken from Barabási, A.-L. (2016). *Network Science*. Cambridge University Press.

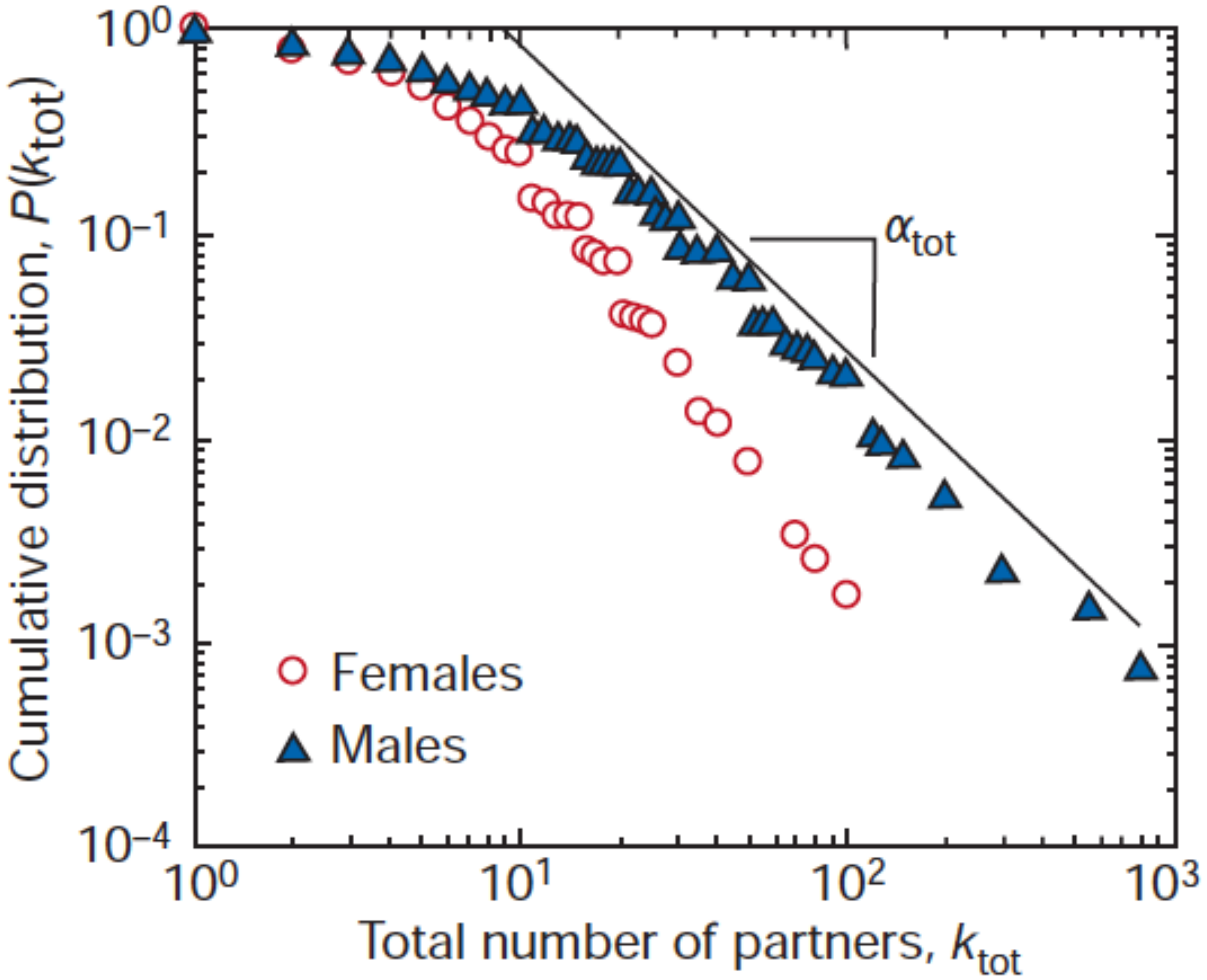
# Absence of eradication threshold sexually transmitted diseases



**scale free network** $P(k) = Ck^{-\gamma}$
the degree distribution decays linearly in a log-log plot

Pastor Satorras and Vespignani (2001) + Lijeros *et al.* (2001)

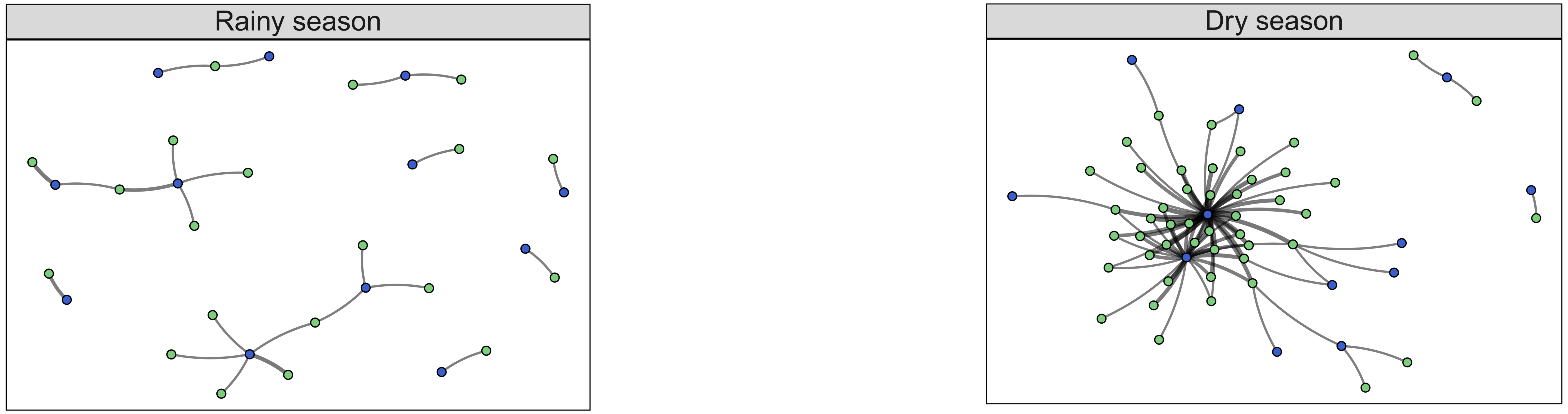


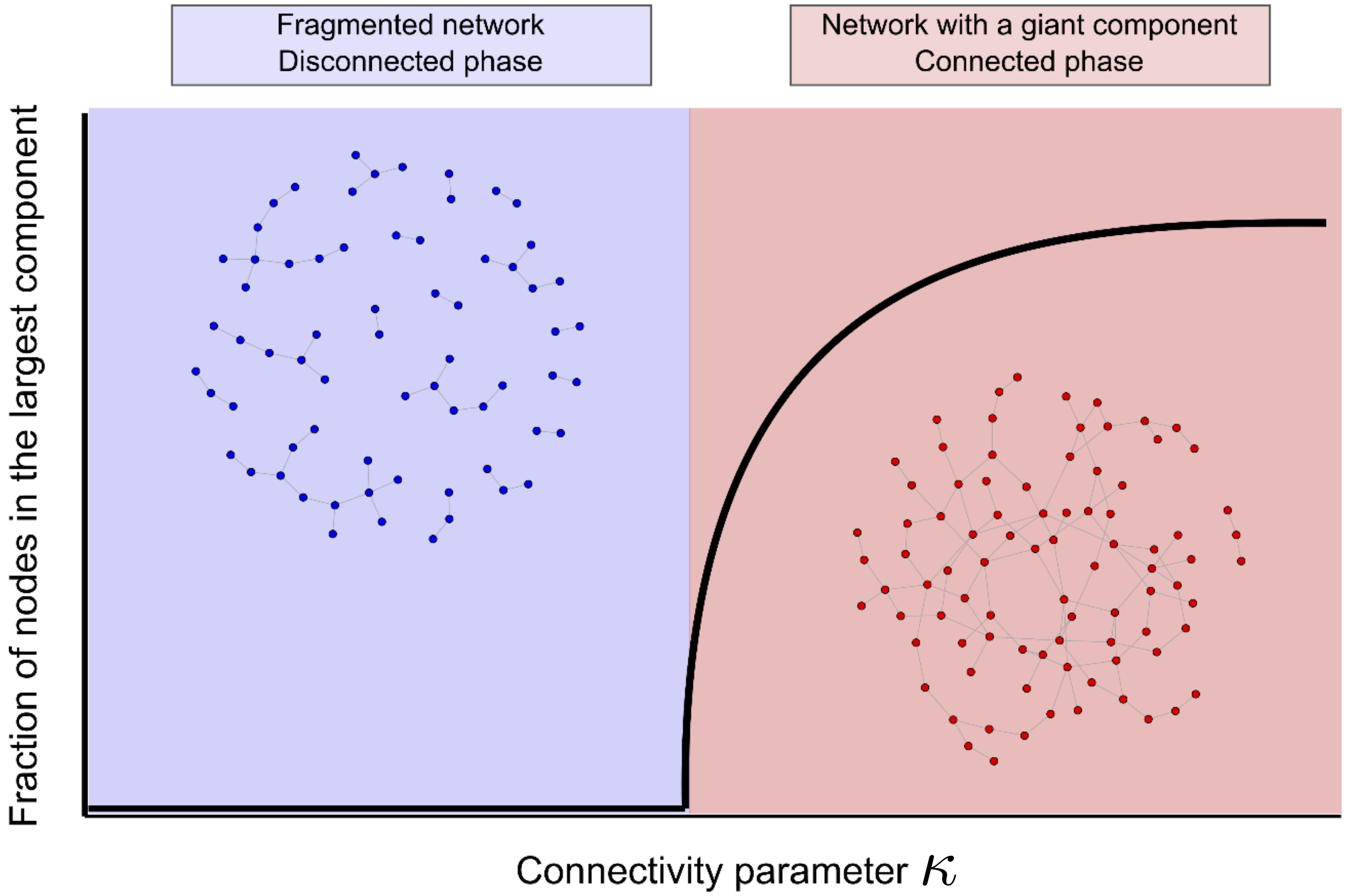



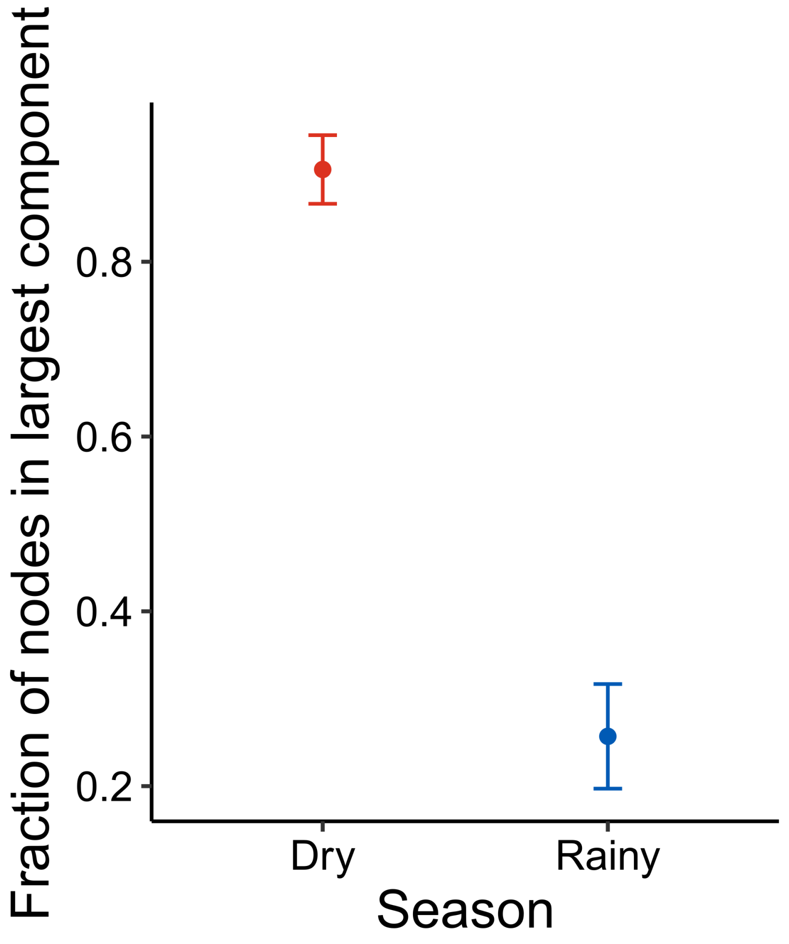
what is relevant for ecology…
…according to a physicist

# Critical transition in ecological networks

driven by seasonality

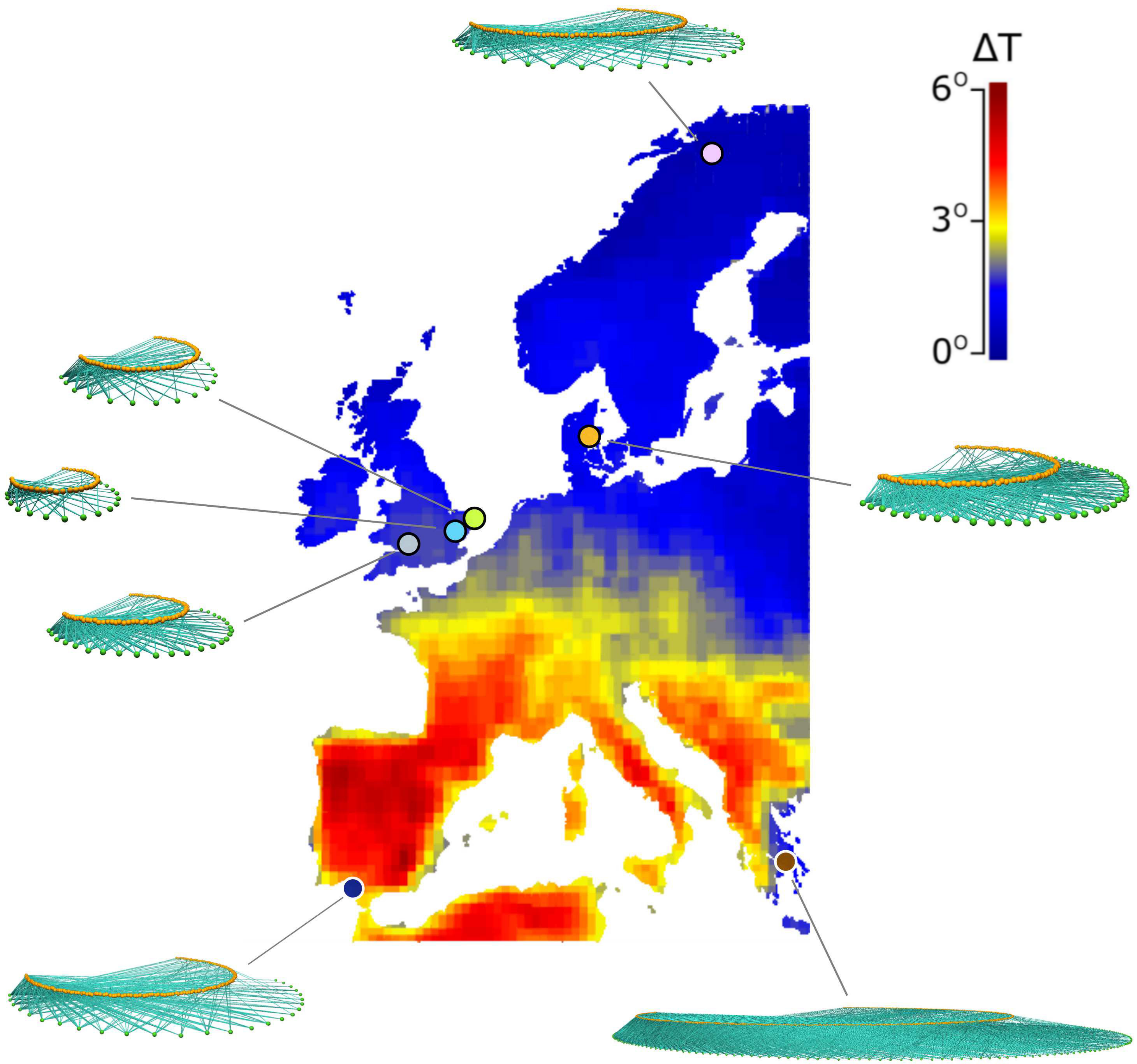


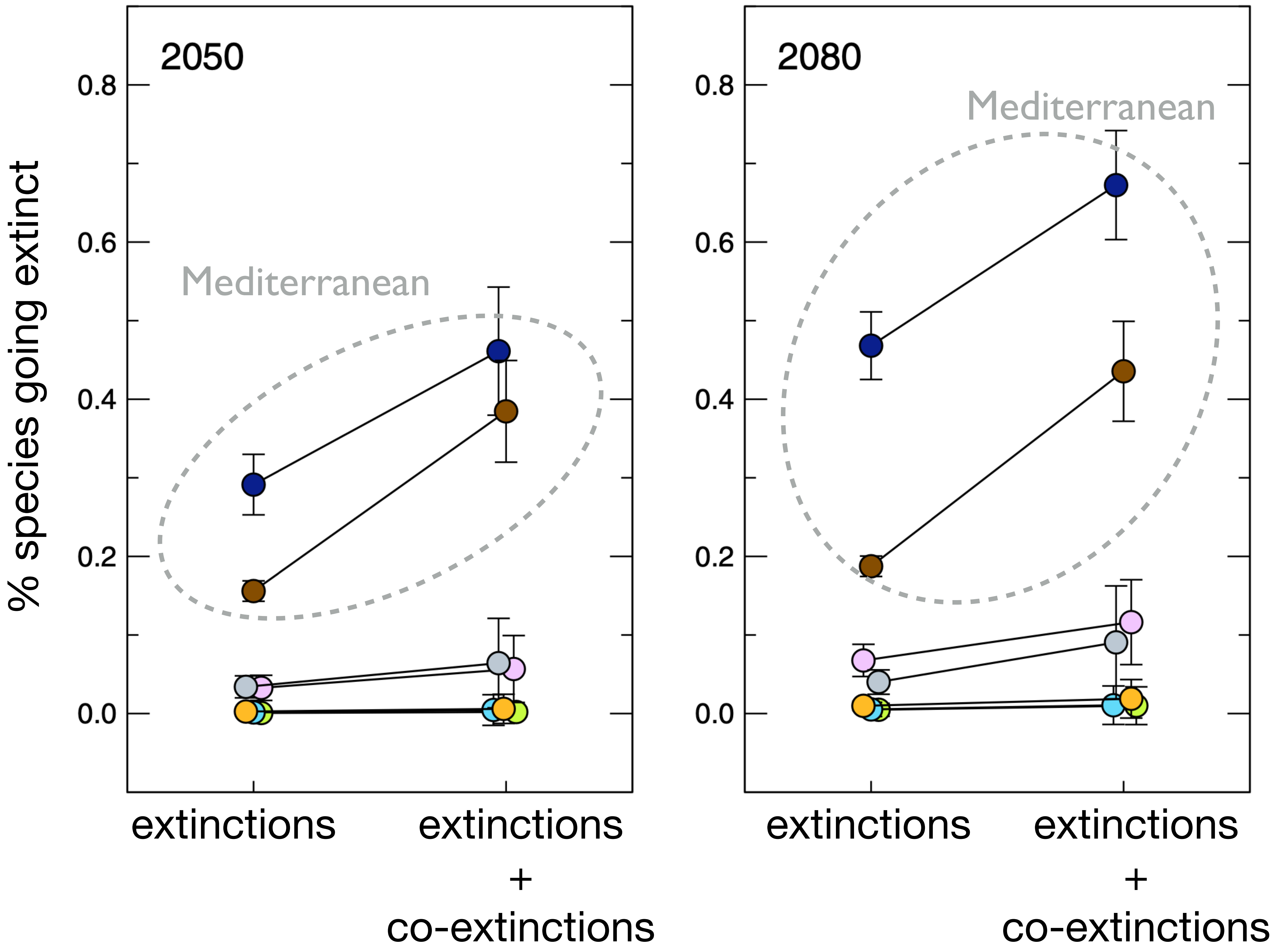




Cosmo, L. G. et al. *manuscript in preparation*

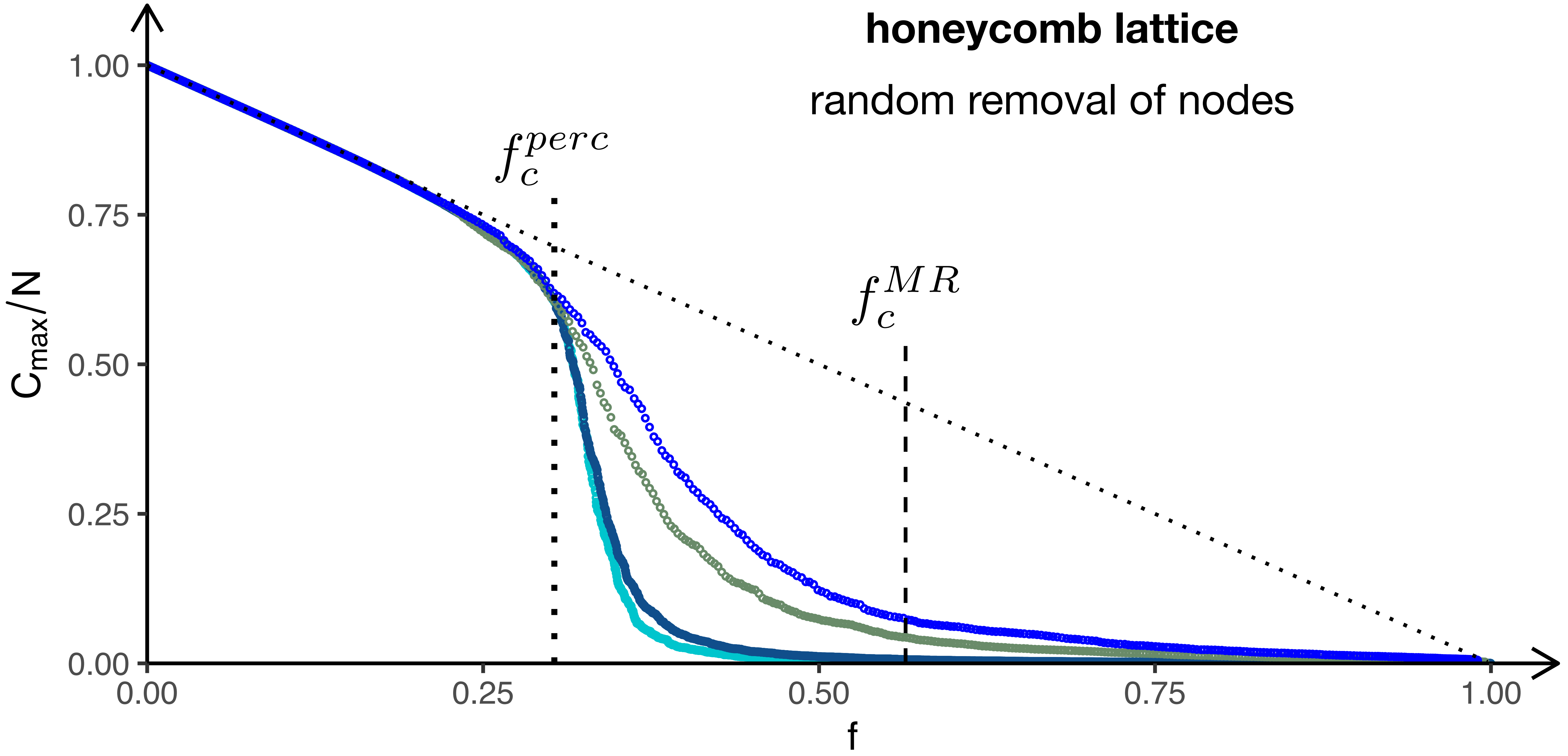
# ecologically-driven removal of nodes

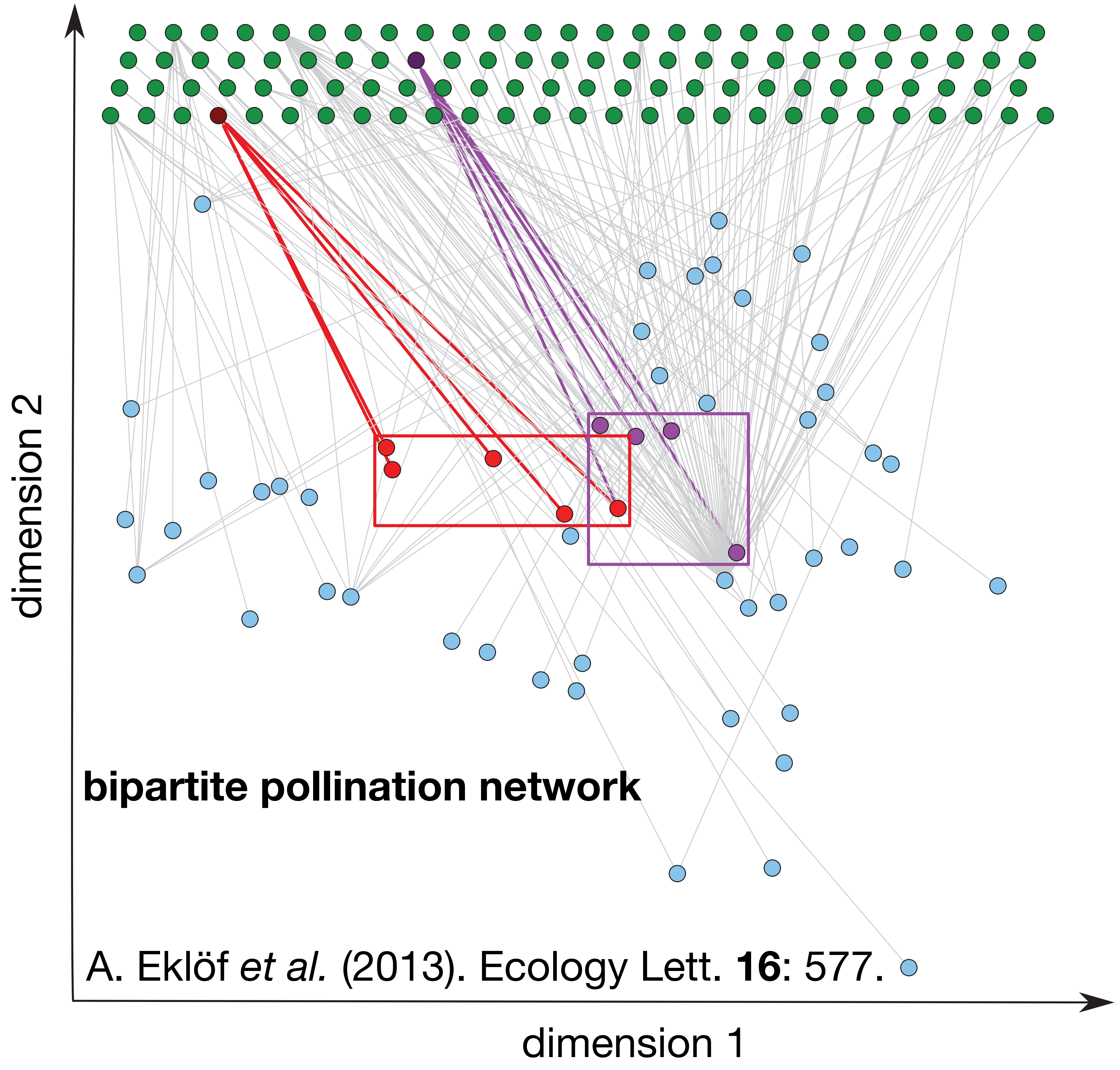



Bascompte, García, Ortega, Rezende, and Pironon (2019). *Sci. Adv.* 5: eaav2539

## 1. are ecological networks really random?




## 2. what role play its finite size or the sampling efforts on assessing the robustness of an ecological network?

# Thank you!

# can we learn something about habitat restoration from physics?

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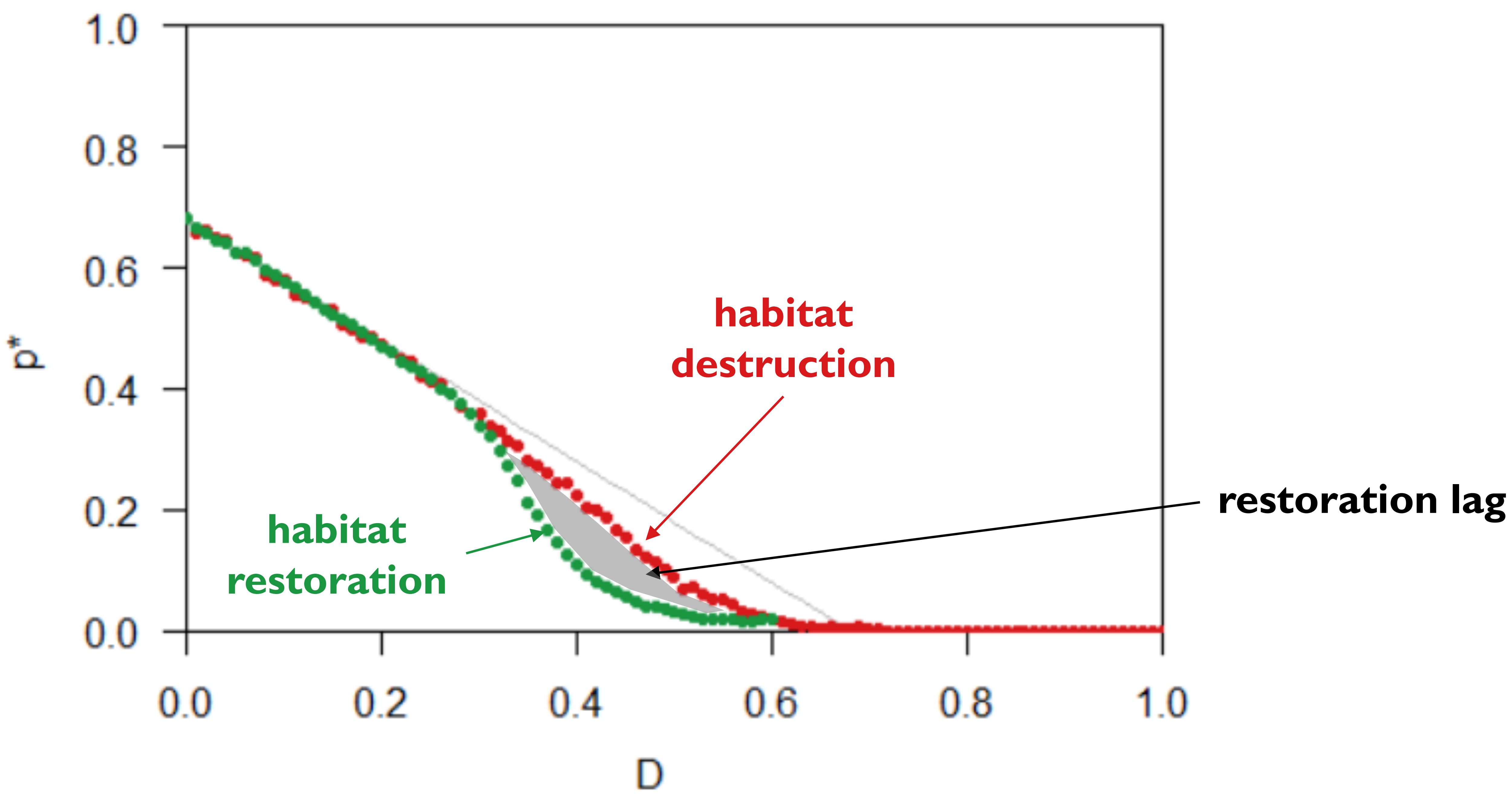
DOI: 10.1111/1365-2656.13450

RESEARCH ARTICLE

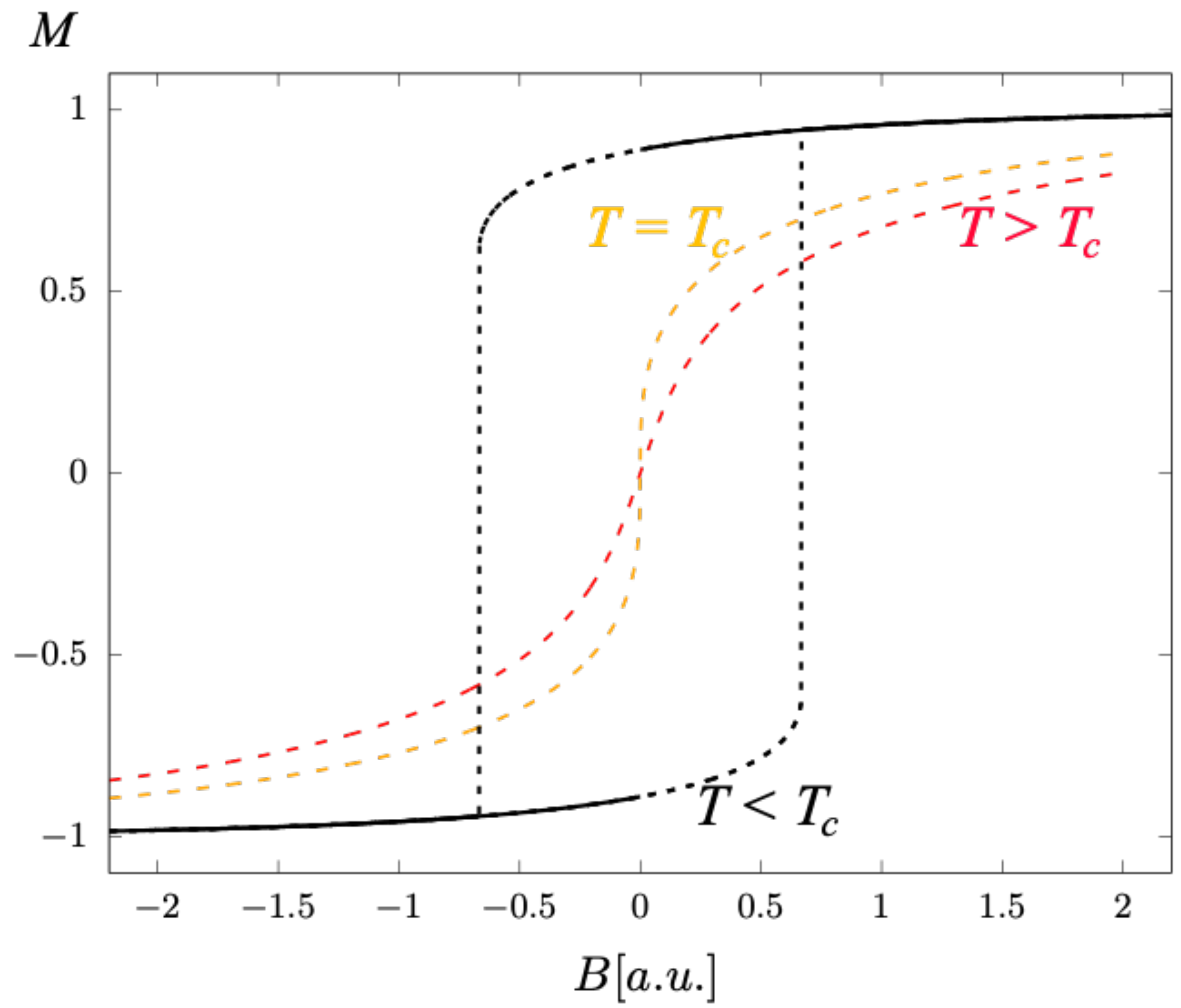
Journal of Animal Ecology | BRITISH ECOLOGICAL SOCIETY

## Habitat restoration in spatially explicit metacommunity models

Klementyna A. Gawecka | Jordi Bascompte




magnetic hysteresis appears in magnets in the **supercritical** (ferromagnetic) phase



## Nobel prizes related to phase transitions

| | |
|---|---|
| 1910 | Johannes Diderik van der Waals |
| 1962 | Lev Davidovich Landau |
| 1968 | Lars Onsager |
| 1977 | Philip Warren Anderson, Nevill Francis Mott, John Hasbrouck Van Vleck |
| 1982 | Kenneth G. Wilson |
| 1991 | Pierre-Gilles de Gennes |
| 2001 | Eric Allin Cornell, Carl Edwin Wieman, Wolfgang Ketterle |
| 2016 | David J. Thouless, John M. Kosterlitz, F. Duncan M. Haldane |
| 2021 | Giorgio Parisi, Klaus Hasselmann, Syukuro Manabe |